Topological Entropy of the Horseshoe map

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Dynamical Systems

$$(X, f)$$

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$$f : X \to X$$

Orbits

Definition 1.1. An *orbit* of $x_0 \in X$ is the set:

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Metric

Let (X, d) be a compact metric space, and $f\colon X \to X$ a continuous map. For each $n \in \mathbb{N}$, the function

$$d_n(x, y) = \max_{0 \le k \le n-1} d(f^k(x), f^k(y))$$

Spanning sets

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$\text{span}(n, \varepsilon, f)$ is the minimum cardinality of any (n, ε) spanning set.

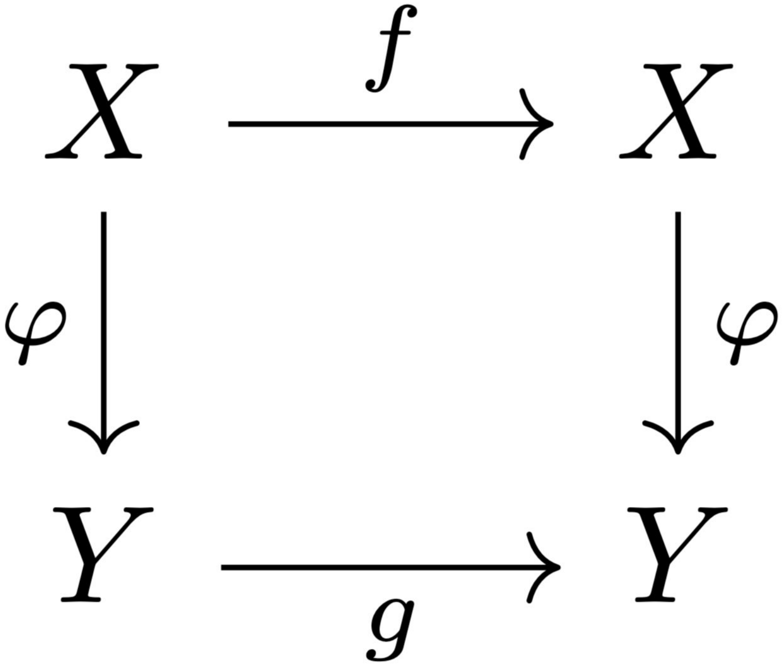
Topological Entropy

$$h(f) = \lim_{\epsilon \to 0^+} \overline{\lim_{n \to \infty}} \frac{1}{n} \log(\mathrm{span}(n, \epsilon, f))$$

Conjugacy

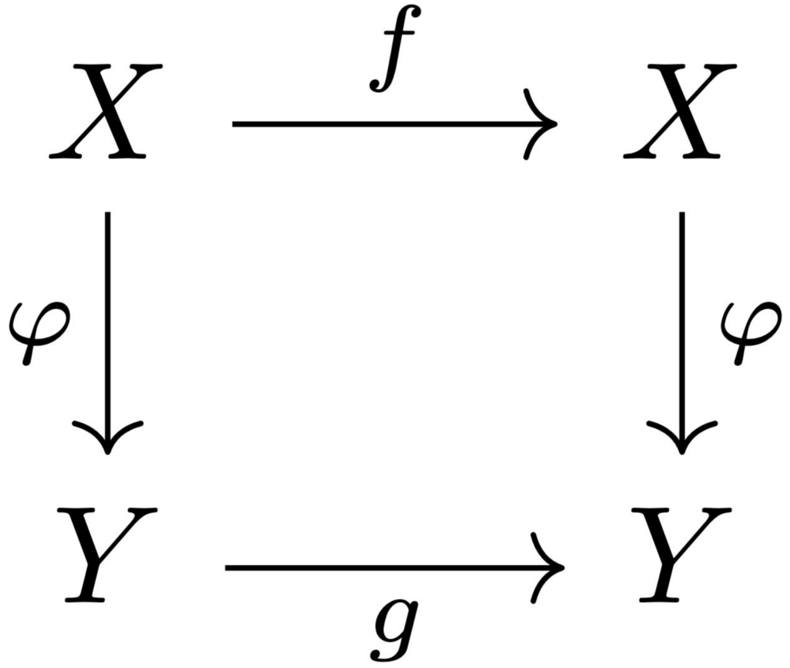
$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & X \\
\varphi \downarrow & & \downarrow \varphi \\
Y & \xrightarrow[\;g\;]{} & Y
\end{array}
$$

Conjugacy



$$\varphi \circ f = g \circ \varphi$$

Conjugacy



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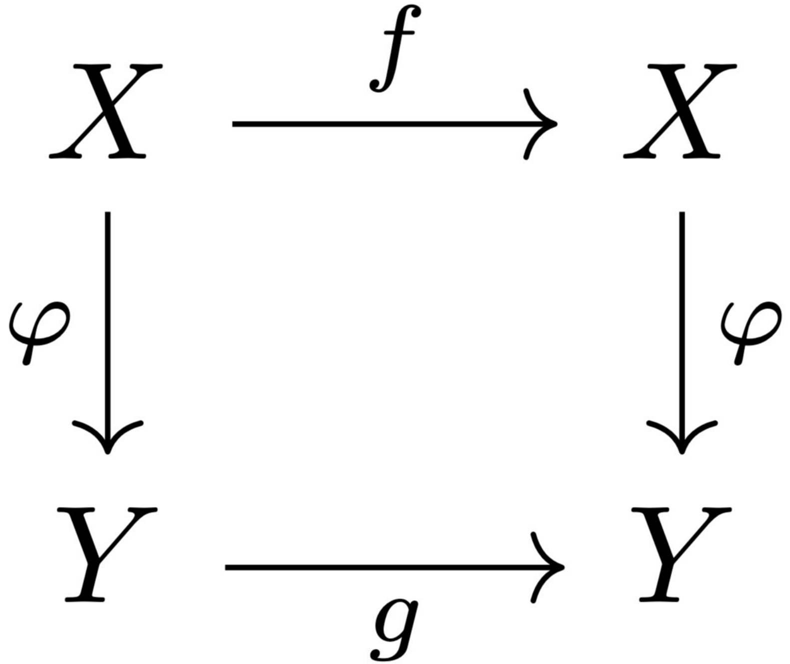
Conjugacy



$$\varphi \circ f = g \circ \varphi$$

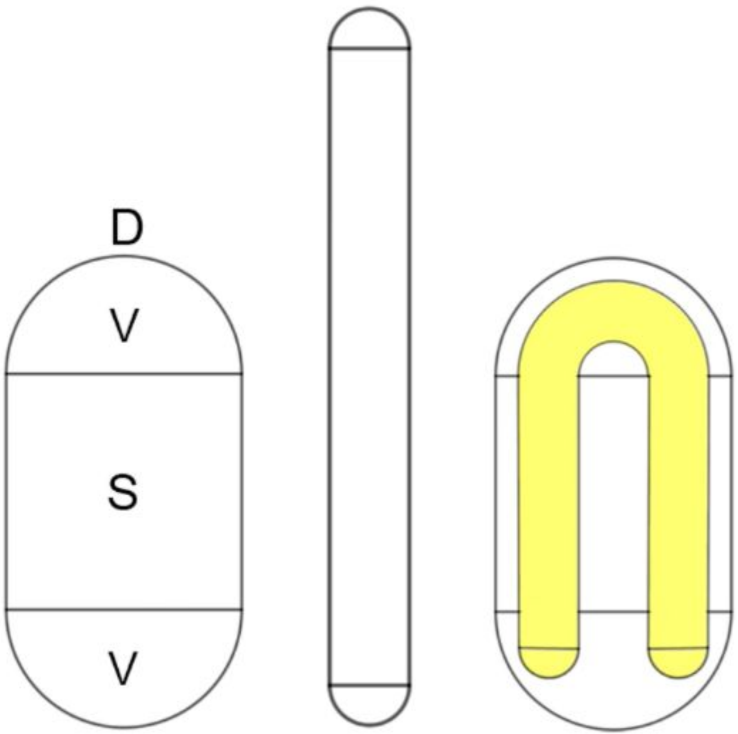
ϕ is a continuous bijection

ϕ^{-1} is continuous

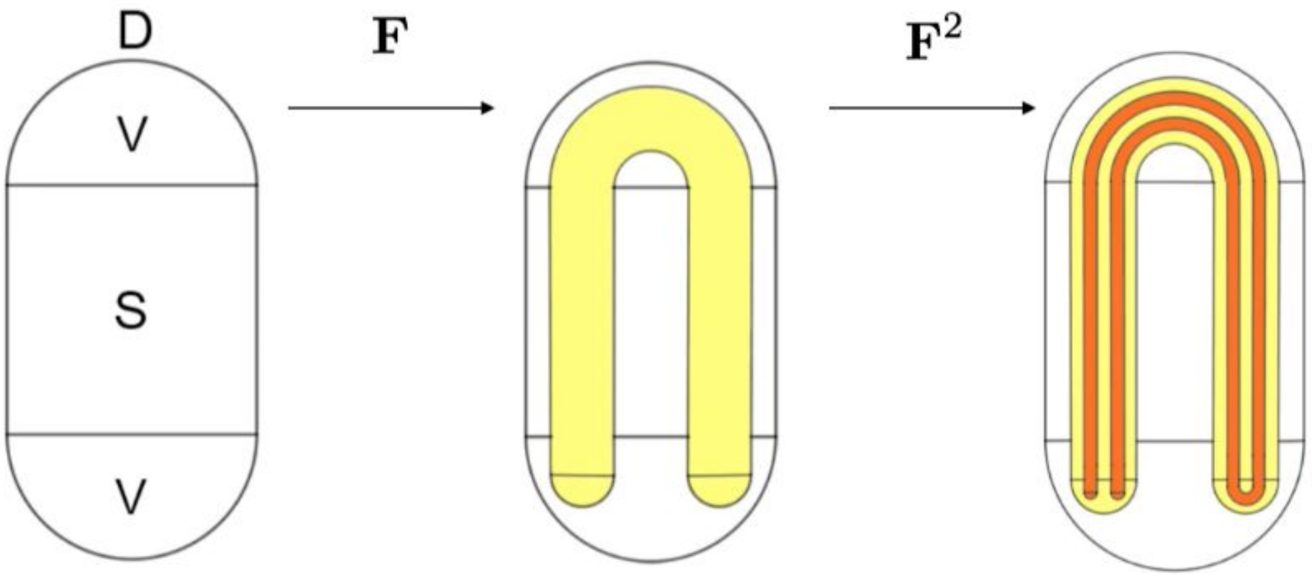
Entropy is invariant under conjugacy [2]

Theorem 5.4. *If* (X, f) *and* (Y, g) *are two topologically conjugate dynamical systems with conjugacy* $\phi : Y \to X$ *then* $h(f) = h(g)$.

Horseshoe [1]



Horseshoe



Horseshoe is conjugate to the Full Shift

Full Shift

$$z = \ldots 100101.101000 \ldots$$

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$$\sigma(z) = \ldots 1001011.01000 \ldots$$

$$\sigma^{-1}(z) = \ldots 10010.1101000 \ldots$$

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Given $\varepsilon > 0, n \in \mathbb{N}$, we want to find our minimum (n, ε) spanning set A.

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Let $x \in \{0,1\}^{\mathbb{Z}}$. To find y such that $d_n(x, y) < \varepsilon$:

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$$y = 011010.110100$$

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$A = \{x^{(w)} : w \in \{0,1\}^{n+2m}\}$

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$$|A| = 2^{n+2m}$$

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$$\lim_{m \to \infty} \limsup_{n \to \infty} \left(1 + \frac{2m}{n}\right) \log(2)$$

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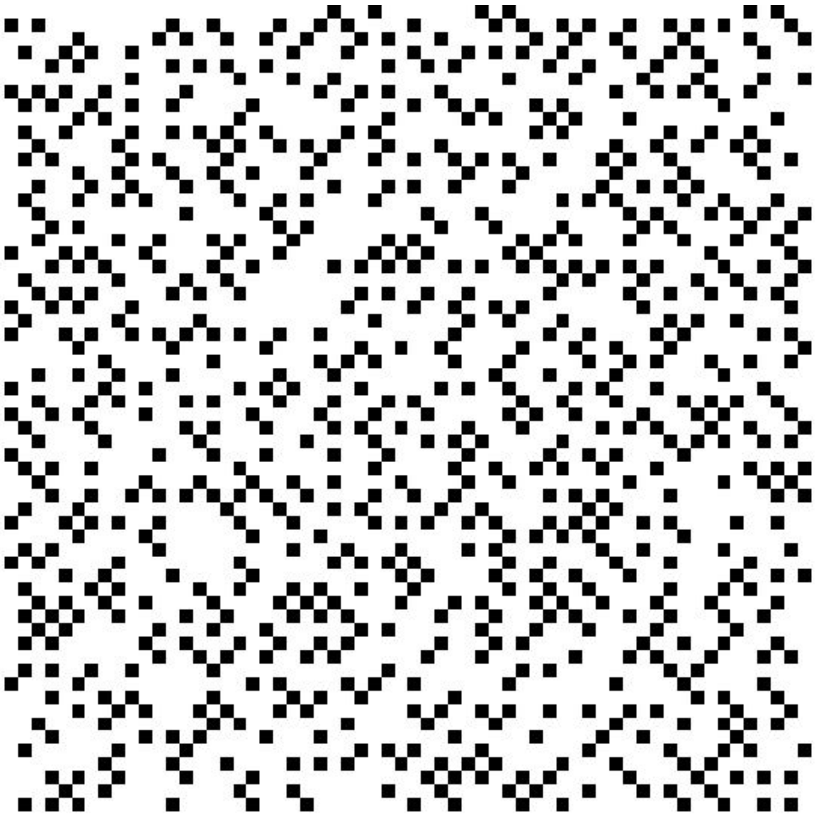
Open Question:

$$\{0,1\}^{\mathbb{Z}^2}$$



Citations

[1] "MA30060 Dynamics and Chaos: the Smale Horseshoe Map." *YouTube*, https://www.youtube.com/watch?v=Ykv1_z6jT3g

[2] Butt, K. (2014). *An introduction to topological entropy* [REU paper]. University of Chicago Mathematics REU. https://math.uchicago.edu/~may/REU2014/REUPapers/Butt.pdf

Thank you!