# Newtonian Mechanics

## Newton's Law

$$F(t, x(t), \dot{x}(t)) = \ddot{x}(t)$$

## Picard-Lindelöf Thm

$\ddot{x} = G(t, x, \dot{x})$ for $G$ differentiable, then $\forall x_0, v_0$, $\exists T > 0$ and a unique solution such that $x(t): (-T, T) \to \mathbb{R}$ solves our ODE

### Method one: Taylor series

for analytic $G$, use

$$x(t) = \sum_{j=0}^{\infty} \frac{a_j}{j!} t^j \quad \text{as solution}$$

$$= \underset{x_0}{\underset{\shortparallel}{a_0}} + \underset{v_0}{\underset{\shortparallel}{a_1}} t + \frac{a_2}{2} t^2 + \frac{a_3}{6} t^3 + \cdots$$

### Method two: Euler-Peano iteration

Let $x(h) \approx x(0) + x'(0)$

$x(2h) \approx x(h) + x'(h)h$

$x(3h) \approx x(2h) + x'(2h)h$

$\cdots\cdots$

### Method three: Picard iteration

Let $x_0(t) = x(0)$

$x_1(t) = x_0(0) + \int_0^t G(s, x_0(s))\, ds$

$x_2(t) = x_1(0) + \int_0^t G(s, x_1(s))\, ds$

$\cdots\cdots$

## Conservation of Energy

$F = -V'(x)$

Given $F(x)$: 1) Calculate $V(x) = -\int F(x)\, dx$, $E = \frac{1}{2}\dot{x}^2 + V(x)$

2) $\displaystyle \Sigma(x) = \int_{x_0}^{x} \frac{1}{\sqrt{2(E_0 - V(x'))}}\, dx'$

3) inverse $\Sigma$ to get: $x(t) = \Sigma^{-1}(\pm t)$

## Equilibrium Points

An equilibrium point $(x_0, 0) \in \mathbb{R}^2$ satisfies:

$F(x) = 0$ $(\nabla V(x) = 0)$ is stable if: $V''(x_0) > 0$, is unstable if: $V''(x_0) < 0$

## Phase portrait

Given a potential $V(x)$, we can plot the phase portrait by:

- Finding all equilibrium points
- Determining their stability

## Conservation of Energy

If $\exists V \in C^2$ s.t. $F = -\nabla V$, then

## Special Case of Poincare Lemma

$\nabla \times F = 0$, i.e. $\partial_j F_i = \partial_i F_j \quad \forall i, j \iff \exists$ scalar function $V \in C^2$ s.t. $F = -\nabla V$

If this holds, $F$ is conservative, and $E = \frac{1}{2}\|\dot{x}\|^2 + V(x)$ is conserved

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## Bound & Unbound motion

A trajectory $x : \mathbb{R}_t \to \mathbb{R}^N$ is bound if $\exists R > 0$ s.t. $\|x(t)\| < R \quad \forall t \in \mathbb{R}$

is unbound otherwise

## Bound motion Proposition

Suppose $F : \mathbb{R}^N \to \mathbb{R}^N$ conservative, with potential $V$ s.t. $\forall \varepsilon > 0, \exists R > 0$ s.t. $|V| < \varepsilon \quad \forall x$ s.t. $\|x\| > R$

then if $E_0 < 0$, the motion is bound

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# Lagrangian Mechanics

## Lagrangian

The Lagrangian $L = T - V$

where $T$ is the kinetic energy of the system and $V$ is the potential energy

## Stationary action

$\gamma$ is a stationary point of the action $\iff$ Euler-Lagrange equations hold:

$$L_x(\gamma, \dot{\gamma}, t) = \frac{d}{dt} L_v(\gamma, \dot{\gamma}, t)$$

## Noether's Thm

Let $\{T^a : x \mapsto x + a w(x) + O(a^2)\}_{a \in \mathbb{R}}$ be a continuous one parameter family where $w(x) = \frac{d}{da} T^a(x) \big|_{a=0}$

s.t. $L(T^a(x), DT^a(x) v, t) = L(x, v, t)$, Then $p := \sum_{j=1}^{N} L_{v_j} w_j$ is conserved

## Conservation of Hamiltonian

If $L$ does not depend on time, then the Hamiltonian $H$ is conserved.

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# Hamiltonian Mechanics

## Legendre Transform

Given Lagrangian $L$, for $p_j = L_{v_j}$, $q_j = x_j$ want map s.t. $X(q,p)$, $v(q,p)$. Not possible if $L_{v_j} \equiv 0 \Rightarrow p_j = 0$

## Hamiltonian

The Hamiltonian $H(q,p) = \sum_{j=1}^{N} p_j \dot{q}_j - L(x(q,p), v(q,p))$

## Hamilton's equations

If $\gamma(t)$ solves EL, then
$q(\gamma(t), \dot\gamma(t))$, $p(\gamma(t), \dot\gamma(t))$ evolve according to:

$$\begin{cases} \dot q_k = \dfrac{\partial H}{\partial p_k} \\ \dot p_k = \dfrac{-\partial H}{\partial q_k} \end{cases}$$

## Hamiltonian Flow

Hamiltonian $H$ generates a flow in phase space $\mathbb{R}^{2N}_{x,p}$,
where $(q,p)$ evolve along an integral curve of the vector field $\begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix}$

## Liouville's Thm

Let $\Phi_t : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$, $(q_0, p_0) \mapsto (q(t), p(t))$

$\Phi_{t+s} = \Phi_s \circ \Phi_t \Rightarrow \Phi_t$ is volume preserving

## Applications in Stat Mech

- $\rho : \mathbb{R}^{2N} \to \mathbb{R}$ is a density if (i) $\rho \geq 0$ everywhere (ii) $\int_{\mathbb{R}^{2N}} \rho(q,p)\, dq\, dp = 1$
- $\rho \circ \Phi_t$ is a density
- The flow $\Phi_t$ generated by a vector field $V$ is volume preserving $\iff \vec\nabla \cdot V = 0$

## Poisson brackets

Let $a(q,p) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be an observable, to measure the rate of change in time of $a$ under evolution of $H$

- $\{a, H\} = \frac{d}{dt} a(q,p) = \sum_{i=1}^{n} \partial_{q_i} a\, \partial_{p_i} H - \partial_{p_i} a\, \partial_{q_i} H$
- $\{c_1 f_1 + c_2 f_2, g\} = c_1 \{f_1, g\} + c_2 \{f_2, g\}$
- $\{f, g\} = -\{g, f\}$
- $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$

Cor: if $\{f, H\} = 0$, $\{g, H\} = 0$, then $\{\{f,g\}, H\} = 0$

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Formula page

- trig integral
- Spherical Coordinate
- Symmetry groups

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