Harmonic functions

Harmonic functions Let $u: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ s.t. $u \in C^2$

say u is harmonic on D if $u_{xx} + u_{yy} = 0$ on D

Harmonic and Analytic Thm

If $u+iv$ is analytic on D, then u and v are harmonic on D

Harmonic Conjugate

Let $u: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be harmonic, the harmonic conjugate $v: D \to \mathbb{R}$ is a harmonic function

for which $f = u+iv$ is analytic on D

Maybe tested!

Existence Thm

Let u be harmonic on D, if D is simply connected, then $\exists$ a harmonic conjugate v on D

Polygonally-connected

$E \subseteq \mathbb{C}$ is polygonally-connected if any two points in E can be connected by a sequence of line segments lying inside the set

Simply connected

A polygonally-connected region D is simple-connected if:

the region enclosed by any simple closed curve is contained in D

Green's Thm If $D \subseteq \mathbb{R}^2$ is a bounded region with positively oriented boundary ∂D

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$

Complex Integration

Line Integrals

Scalar line integral: $\int_\gamma f(x,y)\,ds = \int_a^b f(\gamma(t))\,\|\gamma'(t)\|\,dt$

vector line integral: $\int_\gamma \vec{F}\cdot d\vec{r} = \int_a^b F(\gamma(t))\cdot\gamma'(t)\,dt$

complex line integral: $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$

ML estimate

let γ be piecewise smooth, f continuous on γ, then $\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,|dz|$

If $L = |\gamma(t)|$, $|f(z)| \leq M$ on γ, then $\left|\int_\gamma f(z)\,dz\right| \leq ML$

FTC

I: Let f be continuous on $D \subseteq \mathbb{C}$, $F'(z) = f(z)$ on D,

then for γ w/ $\gamma(0) = A$, $\gamma(1) = B$ $\int_\gamma f(z)dz = \int_A^B f(z)\,dz = F(B) - F(A)$

II: let $D \subseteq \mathbb{C}$ be simply connected, f analytic on D, then f has a primitive F on D

$F(z) = \int_{z_0}^{z} f(w)\,dw$ for any contour on D from z_0 to z

Cauchy's Thm

Let D be a bounded domain with ∂D a finite number of disjoint contours.

If f is analytic on $\partial D \cup D$, then $\int_{\partial D} f(z)\,dz = 0$

Morera's Thm

Let f be continuous on a domain D. If $\int_c f(z)\,dz = 0$ for every closed
- rectangle w/ sides parallel to axis
- Contour
- triangle

in D, then f is analytic on D

Cauchy Integral Formula

Let D be bounded w/ ∂D a contour, if f is analytic on $\partial D \cup D$,

Then $f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z - z_0}\,d$, $z_0 \in D$

Generalized formula: $f^{(m)}(z_0) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(z)}{(z-z_0)^{m+1}}\,dz$ $z_0 \in D$, $m \geq 0$

Cauchy's Estimates

Suppose f is analytic for $|z - z_0| \leq \rho$

if $\exists M > 0$ s.t. $|f(z)| \leq M$ $\forall z$ w/ $|z - z_0| \leq \rho$, then $|f^{(m)}(z_0)| \leq \frac{m!}{\rho^m} M$

Liouville's Thm

If $f(z)$ is entire and bounded, the $f(z)$ is constant