Complex numbers and functions

Complex number

- $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$ • $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$ • $|z|^2 = z\bar{z}$
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta \quad (\theta \in \mathbb{R})$
- $i^n = \begin{cases} 1 & n \equiv 0 \mod 4 \\ i & n \equiv 1 \mod 4 \\ -1 & n \equiv 2 \mod 4 \\ -i & n \equiv 3 \mod 4 \end{cases}$

De Moivre's formula

$\forall n \in \mathbb{N}$:

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = \cos n\theta + i\sin n\theta$$

Roots of unity

for the nth roots of unity of $z^n = w$, $z, w \in \mathbb{C}$

we have n solutions: $z_k = |w|^{1/n} e^{i(\theta + 2\pi k)/n}$ for $k = 0, 1, \dots, n-1$, $\theta = \operatorname{Arg} w$

Complex exponentials

$\exp: \mathbb{C} \to \mathbb{C}$ $\quad e^z = e^x(\cos y + i\sin y)$

Complex logarithm

$$\log z = \ln|z| + i\arg z$$

Complex powers

$$z^a := e^{a\log z} = e^{a(\ln|z| + i\arg z)}$$

- $a \in \mathbb{Q}$, $a = \frac{m}{n}$, $m, n \in \mathbb{Z}$, then z^a takes n distinct values
- $a \in \mathbb{C}$ and $a \notin \mathbb{Q}$, then z^a takes infinitely many distinct values

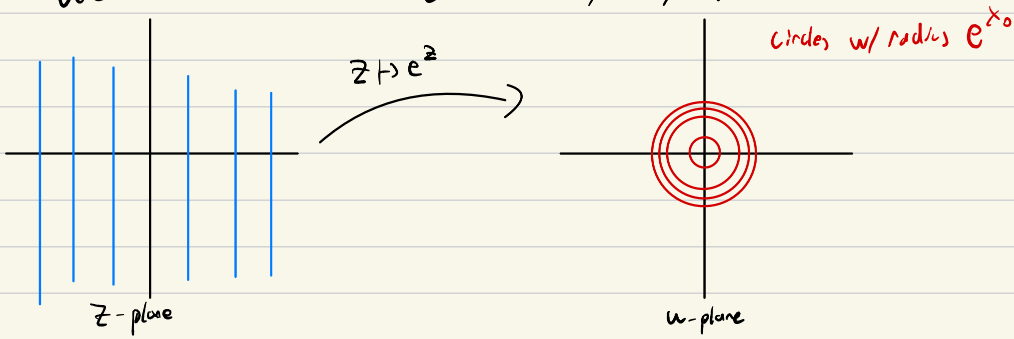
Trigonometric & hyperbolic functions

$\cos z := \dfrac{e^{iz} + e^{-iz}}{2}$, $\quad \sin z := \dfrac{e^{iz} - e^{-iz}}{2i}$, both 2π periodic,

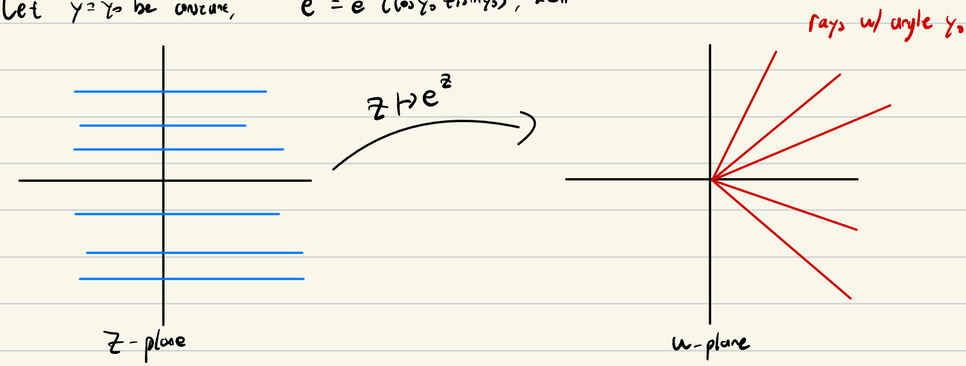
$\cosh z := \dfrac{e^{z} + e^{-z}}{2}$, $\quad \sinh z := \dfrac{e^{z} - e^{-z}}{2}$ $\quad$ $\cos z$, $\sin z$ unbounded on $\mathbb{C}$

Images of exp

Let $x = x_0$ be constant: $e^z = e^{x_0}(\cos y + i\sin y)$ $y \in \mathbb{R}$

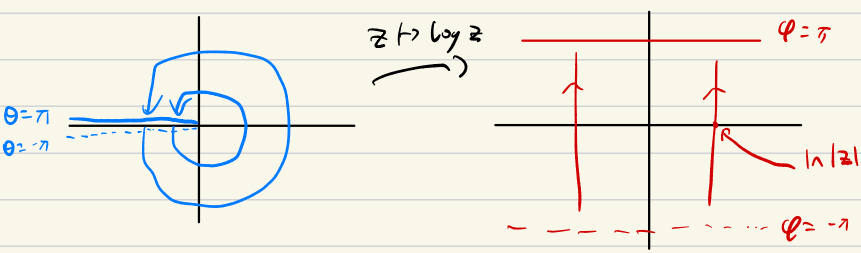


Let $y = y_0$ be constant, $e^z = e^x(\cos y_0 + i\sin y_0)$, $x \in \mathbb{R}$



Images of Log

Log z transforms:
- circles → vertical lines
- rays → horizontal lines



Complex Differeneration

Analytic $f: \mathbb{C} \to \mathbb{C}$ is Analytic on an open $U \subseteq \mathbb{C}$ if
f is $\mathbb{C}$-diff'ble $\forall u \in U$ and f' is continuous $\forall u \in U$

Goursat's Thm If f is $\mathbb{C}$-diff'ble at every point of D, then f' is continuous on D

Cauchy-Riemann Equation $f(z) = u + iv$ on $D \subseteq \mathbb{C}$
f is $\mathbb{C}$-diff'ble at $z \Longrightarrow$ C-R equations hold on z $(u_x = v_y,\ u_y = -v_x)$
$\Longleftarrow$ + 1st partials of u and v are C^1

Constancy Thms Suppose f is analytic on domain $D \subseteq \mathbb{C}$

$(f'(z) = 0\ \forall z \in D)$ or $(f(z) \in \mathbb{R}\ \forall z \in D)$ or (f, $\bar{f}$ analytic on D) $\Rightarrow f$ is constant on D