Complex series

Series convergence

$$\sum_{n=0}^{\infty} a_n \text{ conv} \iff \text{both } \sum_{n=0}^{\infty} Re(a_n) \text{ and } \sum_{n=0}^{\infty} Im(a_n) \text{ conv}$$

Unif conv and analyticity

If $f_n \to f$ uniformly on D and each f_n is analytic on D, then f is analytic on D

Taylor's Thm

Suppose $f(z)$ is analytic for $|z - z_0| < \rho$, then $f(z)$ has power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < \rho) \quad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!}$$

and power series has RoC $= R \geq \rho$. For fixed r s.t. $0 < r < \rho$,

$$a_n = \frac{1}{2\pi i} \oint_{|z - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$$

May need to look again!

Uniqueness principle

If f, g analytic on D and $f(z) = g(z)$ for all $z \in E$ where $E \subseteq D$ contains a nonisolated point, then $f(z) = g(z)$ $\forall z \in D$

Weierstrass M-test

Suppose $M_n \geq 0$ and $\sum M_n$ converges.

If $g_n(z)$ satisfies $|g_n(z)| \leq M_n$, $\forall z \in E$, $\forall n \in \mathbb{N}$, then $\sum_{n=0}^{\infty} g_n(z)$ conv unif and abs on E

Cauchy product formula

For series converging absolutely

$$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum c_n z^n \quad \text{where } c_n = \sum_{j=0}^{n} a_j b_{n-j}$$

Zeros and singularities

Zeros of functions Suppose f is analytic and $\not\equiv 0$ on D

a zero z_0 of f is $z_0 \in D$ s.t. $f(z_0) = 0$

- A zero has order N if $f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$ and $f^{(N)}(z_0) \neq 0$

$\iff f(z) = (z-z_0)^N h(z)$ for some $h(z)$ analytic near z_0 w/ $h(z_0) \neq 0$

Laurent series If f is analytic on an annulus $\rho < |z - z_0| < \sigma$,

then f has a unique Laurent series representation $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

- series converges on subannuli
- analytic part of f is $f_0(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ (where $|z-z_0| < \sigma$)
- principle part of f is $f_1(z) = \sum_{n=-\infty}^{-1} a_n (z-z_0)^n$ (where $|z-z_0| > \rho$)

Isolated Singularities

- z_0 is an isolated singularity of f if f is analytic on $D(z_0, r) \setminus \{z_0\}$ but not at z_0

three cases:
- z_0 is removable if $a_n = 0$ $\forall n < 0$

  $\iff \tilde{f} = \begin{cases} f(z) & z \neq z_0 \\ a_0 & z = z_0 \end{cases}$ is an analytic extension of f

  $\iff |f|$ is bounded near z_0

  $\iff \lim_{z \to z_0} f(z)$ exists
- z_0 is a pole of order N if $a_n = 0$ $\forall n < -N$ and $a_{-N} \neq 0$

  $\iff f(z) = \frac{g(z)}{(z-z_0)^N}$ for some g analytic and nonzero at z_0

  $\iff z_0$ is a zero of order N for $\frac{1}{f}$

  $\iff \lim_{z \to z_0} |f(z)| = \infty$
- z_0 is an essential singularity if $a_n \neq 0$ for infinitely many $n < 0$

  $\iff \exists z_n, z_n' \to z_0$ s.t. $f(z_n) \to L$, $f(z_n') \to M$ and $M \neq L$

Quotient-pole lemma Suppose f, g analytic in some disk centered at z_0. If f has a zero of order n at z_0 (letting $n=0$ if $f(z_0) \neq 0$), and g has a zero of order m at z_0, then

$$h(z) = \frac{f(z)}{g(z)} \text{ has a } \begin{cases} \text{Removable singularity at } z_0 & \text{if } m \leq n \\ \text{pole of order } (m-n) \text{ at } z_0 & \text{if } m > n \end{cases}$$

Residue

Residue Suppose f has Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ on $0 < |z-z_0| < R$

The residue at z_0 is $\text{Res}[f(z), z_0] = a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)dz \quad (0 < r < R)$

Residue Thm Let D be a domain w/ piecewise smooth boundary ∂D. Suppose f is analytic on $D \cup \partial D$ except at a finite number of isolated singularities $z_1, \dots, z_m \in D$, then $\int_{\partial D} f(z) = 2\pi i \sum_{j=1}^{m} \text{Res}[f(z), z_j]$

Rules of residue

Rule 0: If f has a removable singularity at z_0, then $\text{Res}[f(z), z_0] = 0$

Rule 1: If f has a pole of order 1 at z_0, then $\text{Res}[f(z), z_0] = \lim_{z \to z_0} [(z-z_0) f(z)]$

Rule 2: If f has a pole of order N at z_0, then $\text{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} [(z-z_0)^N f(z)]$

Rule 3: If f and g are both analytic near z_0 and g has a simple zero at z_0, then $\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$

Application of Residue

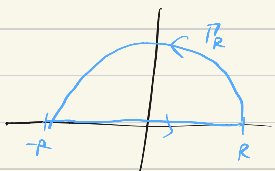
For $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ where P, Q are polynomials and $\deg Q \geq \deg P + 2$,

Then $$\int_{\partial D_R} \frac{P(z)}{Q(z)} dz = \int_{-R}^{R} \frac{P(x)}{Q(x)} dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz$$

$\downarrow$ Evaluate using Residue Theory

$\downarrow$ $R \to \infty$ to get original integral

$\downarrow$ goes to 0 by ML estimate



$\partial D_R = [-R, R] + \Gamma_R$

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