Partial Derivatives

Partial Derivative

$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$ ▪ (2nd-order) $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

C^p functions

Let $f: V \to \mathbb{R}^m$, $p \in \mathbb{N}$,

say f is C^p on V if all partial derivatives of f of order $\leq p$ all exist and are continuous on V

Clairaut's Thm

Let $f: V \to \mathbb{R}^m$, V is open in $\mathbb{R}^n$

if $f \in C^2(V)$, then $\forall a \in V$, $j,k \in \{1, \dots, n\}$, $\frac{\partial^2 f}{\partial x_k \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_k}(a)$

Interchange of limits and integrals

Let $f: H \to \mathbb{R}$ be continuous, where $H = [a,b] \times [c,d] \subseteq \mathbb{R}^2$, $F(y) = \int_a^b f(x,y)\,dx$ for $y \in [c,d]$

Then F is continuous on $[c,d]$, i.e. $\lim_{y \to y_0} \int_a^b f(x,y)\,dx = \int_a^b \lim_{y \to y_0} f(x,y)\,dx$ $\forall y_0 \in [c,d]$

Interchange of Derivative and Integral

Let $f: H \to \mathbb{R}$ be C^1 on H, $H = [a,b] \times [c,d] \subseteq \mathbb{R}^2$, then

$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$, $\forall y \in [c,d]$

Linear Transformations

Notations

- $A \in L(\mathbb{R}^n, \mathbb{R}^m) = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

Norm of Matrix

- Define norm of A by $\|A\| = \sup_{\|x\| \le 1} \|Ax\|$, since A is continuous on $\{x \in \mathbb{R}^n : \|x\| = 1\}$ compact, by EVT $\|A\| < \infty$, $\exists x \in \mathbb{R}^n$ w/ $\|x\| = 1$ s.t. $\|A\| = \|Ax\|$

- $\|A\|$ is the smallest number s.t. $\|Ax\| \le \|A\| \|x\|$ $\forall x \in \mathbb{R}^n$
- $\|A\| \ge 0$, $\|A\| = 0 \iff A = 0$ • $\|\lambda A\| = |\lambda| \|A\|$, $\forall \lambda \in \mathbb{R}$ • $\|A + B\| \le \|A\| + \|B\|$
- $\rho(A, B) = \|A - B\|$ is a metric on $L(\mathbb{R}^n, \mathbb{R}^m)$
- $\|CA\| \le \|C\| \|A\|$ (matrix multiplication)
- $\|A\| \le \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$

Cauchy Schwartz

$|u \cdot v| \le \|u\| \|v\|$

Made with Goodnotes

Differentiability

Total derivative Let $V \subseteq \mathbb{R}^n$ be open, $a \in V$

- f is diff'ble at a $\iff$ all first order partial derivatives of f exist at a

$$B := Df(a) = \left[\frac{\partial f_i}{\partial x_j}(a)\right]_{m \times n} \quad \text{satisfies} \quad \lim_{h \to 0} \frac{\|f(a+h) - f(a) - Bh\|}{\|h\|} = 0$$

Continuously differentiable Let $V \subseteq \mathbb{R}^n$ be open, $a \in V$, $f: V \to \mathbb{R}^m$

- Say f is continuously diff'ble at a if $\exists B_r(a) \subseteq V$ w/ $r > 0$ s.t. all first order partial derivatives exist in $B_r(a)$ and are continuous at a
- Say f is continuously diff'ble on V if $f \in C^1(V)$

Continuously differentiable implies differentiable

f is continuously diff'ble at $a \Rightarrow f$ is diff'ble at a

Chain rule Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open, $f: U \to \mathbb{R}^m$ be diff'ble at $a \in U$, $f(U) \subseteq V$, $g: V \to \mathbb{R}^k$ diff'ble at $f(a)$

Then $g \circ f: U \to \mathbb{R}^k$ is diff'ble at a and $D(g \circ f)(a) = Dg(f(a))Df(a)$

MVT

Line segment Let $x, a \in \mathbb{R}^n$,

the line segment between x and a is $L(x;a) = \{a + t(x-a) : 0 \le t \le 1\}$

Convexity

Say $E \subseteq \mathbb{R}^n$ is convex if $L(x;a) \subseteq E$, $\forall x, a \in E$

Scalar function MVT Let $f: V \to \mathbb{R}$ be diff'ble on V, where $V \subseteq \mathbb{R}^n$ is open

Then $\forall x, a \in V$ w/ $L(x;a) \subseteq V$, $\exists c \in L(x;a)$ s.t.

$$f(x) - f(a) = Df(c)(x-a) = \nabla f(c) \cdot (x-a)$$

Vector-valued MVT Let $f: V \to \mathbb{R}^m$ be diff'ble on V, where $V \subseteq \mathbb{R}^n$ is open

Then $\forall x, a \in V$ w/ $L(x;a) \subseteq V$, $\exists c \in L(x;a)$ s.t.

$$\|f(x) - f(a)\| \le \|Df(c)\| \cdot \|x-a\|$$

Continuity of derivative map Let $V \subseteq \mathbb{R}^n$ be open, $f: V \to \mathbb{R}^m$

if $f \in C'(V)$, then $Df: V \to L(\mathbb{R}^n, \mathbb{R}^m)$, $a \mapsto Df(a)$ is continuous

Bound on compact and convex sets Let $f: V \to \mathbb{R}^m$ be C' on V, where $V \subseteq \mathbb{R}^n$ is open.

Then $\forall$ compact and convex $K \subseteq V$, $\exists M \ge 0$ s.t. $\|f(x) - f(a)\| \le M\|x-a\|$, $\forall x, a \in K$