Compactness

Covers Let $E \subseteq X$

- An open cover of E is $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of open sets V_α in X such that $E \subseteq \bigcup_{\alpha \in A} V_\alpha$
- $\mathcal{V}$ has a finite subcover if $\exists$ a finite set $A_0 \subseteq A$ such that $E \subseteq \bigcup_{\alpha \in A_0} V_\alpha$

Compactness

E is compact $\Rightarrow$ every open cover of E has a finite subcover

Compact and sequentially compact In metric space X, $E \subseteq X$

E is compact $\Longleftrightarrow$ E is sequentially compact

Heine-Borel In metric space X that satisfies BW property, $E \subseteq X$

E is compact $\Longleftrightarrow$ E is closed and bounded

Uniform Continuity

f continuous on compact E $\Rightarrow$ f is uniformly continuous on E

Continuity preserves Compactness

f is continuous on compact E $\Rightarrow$ $f(E)$ is compact in Y

Continuity and Pre-image

f is continuous on E $\Longleftrightarrow$ $f^{-1}(V)$ is open in E, $\forall V$ open in Y
$\Longleftrightarrow$ $f^{-1}(B)$ is closed in E, $\forall B$ closed in Y

Extreme Value Theorem $\emptyset \neq E$, $f: E \to \mathbb{R}$

f continuous on compact E $\Rightarrow$ $\exists a, b \in E$ s.t. $f(a) = \inf_{x \in E} f(x)$, $f(b) = \sup_{x \in E} f(x)$

Continuity of Inverse function

f continuous on compact E, one-to-one $\Rightarrow$ f^{-1} is continuous

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Images and inverse (pre) images

Let X, Y be sets, $f: X \to Y$,

1) Let $B, D, E_\alpha \subseteq Y$, $\alpha \in A$ some index set

- $f^{-1}(\bigcup_{\alpha \in A} E_\alpha) = \{x \in X : f(x) \in E_\alpha \text{ for some } \alpha \in A\} = \bigcup_{\alpha \in A} f^{-1}(E_\alpha)$
- $f^{-1}(\bigcap_{\alpha \in A} E_\alpha) = \{x \in X : f(x) \in E_\alpha, \ \forall \alpha \in A\} = \bigcap_{\alpha \in A} f^{-1}(E_\alpha)$
- $f^{-1}(D \setminus B) = \{x \in X : f(x) \in D \text{ and } f(x) \notin B\} = f^{-1}(D) \setminus f^{-1}(B)$

2) Let $B, D, E_\alpha \subseteq X$, $\alpha \in A$

- $f(\bigcup_{\alpha \in A} E_\alpha) = \{f(x) : x \in E_\alpha \text{ for some } \alpha \in A\} = \bigcup_{\alpha \in A} f(E_\alpha)$
- $f(\bigcap_{\alpha \in A} E_\alpha) \subseteq \bigcap_{\alpha \in A} E_\alpha$
- $f(D \setminus B) \supseteq f(D) \setminus f(B)$

} equality when f is one to one

Density and separable

Let $E \subseteq X$

- $\bar{E} = \{x \in X : \forall r > 0, \ B_r(x) \cap E \neq \emptyset\}$
- E is dense in X if $\bar{E} = X$, i.e. $\forall V \neq \emptyset$ open in X, $V \cap E \neq \emptyset$
- X is separable if it has a countable dense subset

Lindelöf's Thm

Let X be separable, $E \subseteq X$

Every open cover $\{V_\alpha : \alpha \in A\}$ of E has a countable subcover

i.e. $\exists \{\alpha_k : k \in \mathbb{N}\} \subseteq A$ s.t. $E \subseteq \bigcup_{k=1}^{\infty} V_{\alpha_k}$

Subspace Topology

Let X be a metric space, $E \subseteq X$, $U, A \subseteq E$

1) U is open in $E \iff \exists$ an open set V in X s.t. $U = V \cap E$
2) A is closed in $E \iff \exists$ a closed set B in X s.t. $A = B \cap E$

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Connected

Disconnected and Connected

Let (X, ρ), $E \subseteq X$

- E is disconnected if $\exists$ nonempty disjoint sets U and V open in E s.t. $E = U \cup V$
- E is connected if it is not disconnected

  i.e. $E = U \cup V$ w/ U, V disjoint and open in E

  $\Rightarrow U = \phi$ or $V = \phi$

Open and closed in Connected set

E is connected $\Leftrightarrow$ ϕ and E are the only subsets of E that are both open and closed in E

Intervals are connected

in $(\mathbb{R}, d)$

E is connected $\Leftrightarrow$ E is an interval ($a, b \in E$, if $\exists c$ s.t. $a < c < b \Rightarrow c \in E$)

Continuity preserves Connectedness

f is continuous on connected $E \Rightarrow f(E)$ connected

Intermediate Value Thm

$f: E \to \mathbb{R}$

f is continuous on connected E, $f(a) \neq f(b)$ for $a, b \in E$, y s.t. $f(a) < y < f(b)$

$\Rightarrow \exists x \in E$ s.t. $f(x) = y$

Path Connected

E is path connected $\Leftrightarrow$ $\forall x, y \in E$, $\exists$ a continuous function $f: [0,1] \to E$ w/ $f(0) = x$, $f(1) = y$

Path Connected implies Connected

E is path-connected $\Rightarrow$ E is connected

Identity Thm

Let $f, g: I \to \mathbb{R}$ be analytic, where I is an open interval. $E = \{x \in I : f(x) = g(x)\}$

If I contains a cluster point of E, then $f = g$ on I

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Stone-Weierstrass

Uniform Metric space Let X be compact, $C(X) = \{f: X \to \mathbb{R} : f \text{ is cts}\}$

For $f, g \in C(X)$, $\|f\| = \sup_{x \in [a,b]} |f(x)|$, $\rho(f,g) = \|f-g\|$

Then $f_n \to f$ in $C(X) \iff f_n \to f$ uniformly on X

Algebra Let X be a set, $\mathcal{A}$ a set of functions from X to $\mathbb{R}$

1) $\mathcal{A}$ is an algebra if:

$f+g,\ fg,\ \alpha f \in \mathcal{A},\quad \forall f, g \in \mathcal{A},\quad \forall \alpha \in \mathbb{R}$

E.g.

$\mathcal{A} = \{\sum_{k=0}^{n} a_k x^k : a_k \in \mathbb{R},\ n \in \mathbb{N} \cup \{0\}\}$

is the set of polynomials on $\mathbb{R}$

2) $\mathcal{A}$ separates points of X if:

$\forall x, y \in X$ w/ $x \neq y$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$

3) $\mathcal{A}$ vanishes at no point of X if:

$\forall x \in X$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$

4) $\mathcal{A}$ is an algebra $\Rightarrow \bar{\mathcal{A}}$ is an algebra

Stone-Weierstrass Thm Let X be compact, $C(X)$ the uniform metric space, $\mathcal{A} \subseteq C(X)$

$\mathcal{A}$ is
- an algebra
- separates points of X
- vanishes at no point of X

$\implies \bar{\mathcal{A}} = C(X)$, i.e. $\forall f \in C(X)$, $\exists$ a seq $\{p_n\}$ in $\mathcal{A}$ s.t. $p_n \to f$ uniformly on X

Weierstrass Approximation Thm "Special case"

$f \in C[a,b] \Rightarrow \exists$ a sequence of polynomials $p_n \to f$ uniformly on $[a,b]$

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