Inverse Function Thm & Implicit Function Thm

Invertible Let $\Omega = \{A \in L(\mathbb{R}^n) : A \text{ is invertible}\}$

- $A \in \Omega$ and $B \in L(\mathbb{R}^n)$ w/ $\|B - A\| < \frac{1}{2\|A^{-1}\|}$, then $B \in \Omega$ w/ $\|B^{-1}\| \leq 2\|A^{-1}\|$
- $A \mapsto A^{-1}$ is a continuous map

C-Lipschitz $f: X \to X$ is C-Lipschitz for $c < 1$ if $\rho(f(x), f(y)) \leq c\rho(x,y)$ $\forall x, y \in X$

Contraction principle Let (X, ρ) be nonempty and complete metric space. If $f: X \to X$ is C-Lipschitz for $c<1$, then it is a contraction and $\exists! x \in X$ s.t. $f(x) = x$

Inverse Function Thm Let $V \subseteq \mathbb{R}^n$ be open, $a \in V$, $f: V \to \mathbb{R}^n$ C^1 on V, and $Df(a)$ invertible

Then $\exists\, U = B_r(a) \subseteq V$ w/ $r > 0$ s.t.

- f is 1-to-1 on U, • $f(U)$ is open • $f^{-1}: f(U) \to U$ is C^1 on $f(U)$

Moreover, $D(f^{-1})(f(x)) = (Df(x))^{-1}$, $\forall x \in U$

Partial Jacobians Let $f = (f_1, \dots, f_n): V \to \mathbb{R}^n$, where $V \subseteq \mathbb{R}^{n+m}$ is open. $(x, t) \in V$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}^m$

- $D_{f_x} = \left[\frac{\partial f_i}{\partial x_j}\right]_{n \times n}$ • $Df_t = \left[\frac{\partial f_i}{\partial t_j}\right]_{n \times m}$ • $Df = \left[Df_x \quad Df_t\right]_{n \times (n+m)}$

Implicit Function Thm Let $V \subseteq \mathbb{R}^{n+m}$ be open, $(a,b) \in V$ w/ $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $f: V \to \mathbb{R}^n$ C^1 on V, $f(a,b) = 0$, and $Df_x(a,b)$ is invertible.

Then $\exists\, U = B_r(a,b) \subseteq V$ w/ $r > 0$ and open set $W \subseteq \mathbb{R}^m$ w/ $b \in W$ s.t. for each $t \in W$, $\exists!\, x$ w/ $(x,t) \in U$ s.t. $f(x,t) = 0$. Let $g(t) := x$, then $g: W \to \mathbb{R}^n$ is C^1 on W, $g(b) = a$, $f(g(t), t) = 0$, and $Dg(t) = -[Df_x(g(t), t)]^{-1} Df_t(g(t), t)$

Riemann-Lebesgue Thm

Upper and lower sums Let S be subrectangles of P

$$U(f,P) = \sum_{S\in P} M_S(f)\,v(S), \quad M_S(f) = \sup_{x\in S} f(x), \quad L(f,P) = \sum_{S\in P} m_S(f)\,v(S), \quad m_S(f) = \inf_{x\in S} f(x)$$

$$\underline{\int_A} f = \sup_{P\in\mathcal{P}} L(f,P), \quad \overline{\int_A} f = \inf_{P\in\mathcal{P}} U(f,P)$$

Measure zero Let $E \subseteq \mathbb{R}^n$

E has measure zero if $\forall \varepsilon > 0$, $\exists$ a countable cover $\{A_1, A_2, \dots\}$ of E by closed/open rectangles s.t. $\sum_{j=1}^{\infty} v(A_j) < \varepsilon$

Content zero

E has content zero if $\forall \varepsilon > 0$, $\exists$ a finite cover $\{A_1, \dots, A_m\}$ of E by closed/open rectangles s.t. $\sum_{j=1}^{m} v(A_j) < \varepsilon$

Riemann-Lebesgue Thm Let $f: A \to \mathbb{R}$ be bounded, where A is a closed rectangle in $\mathbb{R}^n$

f is integrable on $A \iff D_f = \{x\in A : f \text{ is discontinuous at } x\}$ has measure zero

Jordan measurability and Fubini

Characteristic function Let A be a closed rectangle in $\mathbb{R}^n$, $E \subseteq A$

Let $\chi_E : A \to \mathbb{R}$ be the characteristic function, $\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \in A \setminus E \end{cases}$

Jordan-measurable

Say $E \subseteq \mathbb{R}^n$ is Jordan-measurable if E is bounded and ∂E has measure 0

Integrability of χ_E

- If $E \subseteq A$ is Jordan measurable for a closed rectangle A in $\mathbb{R}^n$, then χ_E is integrable, $\int_A \chi_E = \int_E 1$
- E has content zero $\iff$ E is Jordan-measurable, $E \subseteq A$ for some closed rectangle A, and $\int_A \chi_E = 0$
- E has measure zero, E is Jordan-measurable, $E \subseteq A$ for some closed rectangle A $\Rightarrow \int_A \chi_E = 0$

Fubini's Thm Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be closed rectangles, $f : A \times B \to \mathbb{R}$ be integrable on $A \times B$.

Then
$$\int_{A \times B} f = \int_A \underline{\int_B} f(x,y)\,dy\,dx = \int_A \overline{\int_B} f(x,y)\,dx\,dy$$
$$= \int_B \underline{\int_A} f(x,y)\,dx\,dy = \int_B \overline{\int_A} f(x,y)\,dy\,dx$$

In particular, if for each $x \in A$, $\int_B f(x,y)\,dy$ exists, and for $y \in B$, $\int_A f(x,y)\,dx$ exists,

Then
$$\int_{A \times B} f = \int_A \int_B f(x,y)\,dy\,dx = \int_B \int_A f(x,y)\,dx\,dy$$