Uniform Convergence

Limit Swapping

$g_n \to g$ uniformly and each g_n is continuous at $c \in E$
$\Rightarrow$ g continuous at c

$$\lim_{x\to c}\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} g_n(c) = \lim_{n\to\infty}\lim_{x\to c} g_n(x)$$

Uniform Convergence and differentiation

Let $a<b$ in $\mathbb{R}$, if each f_n diff'ble on (a,b), f_n' converge uniformly on (a,b), $f_n(x_0)$ converge for some $x_0 \in (a,b)$

$\Rightarrow$ f_n converge uniformly and $\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} f_n'$ on (a,b)

Uniform Cauchy Criterion

f_n converges uniformly on E $\iff$ f_n is uniformly Cauchy on E

$\iff$ $\forall \varepsilon>0$, $\exists N \in \mathbb{N}$ s.t. $\forall n,m \geq N$,
$|f_m(x) - f_n(x)| < \varepsilon \quad \forall x \in E$

Weierstrass M-test

$f_k : E \to \mathbb{R}$, For each $k \in \mathbb{N}$, $\exists M_k \in \mathbb{R}$ s.t.

$$|f_k(x)| \leq M_k \quad \forall x \in E, \text{ and } \sum_{k=1}^{\infty} M_k \text{ converges}$$

$\Rightarrow$ $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on E

Swapping series

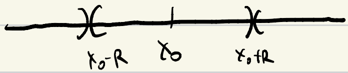
If $\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{kj}$ converges absolutely, then $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{kj} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{kj}$

Power Series

Power Series definition

A power series centered at $x_0 \in \mathbb{R}$ with radius of convergence R

$$f(x) := \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{where } a_k \in \mathbb{R}$$



- $f(x)$ converges absolutely for $|x - x_0| < R$
- $f(x)$ diverges for $|x - x_0| > R$
- $f(x)$ converges uniformly on $[a,b] \subseteq (x_0 - R, x_0 + R)$

$$R := \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} \quad \text{or} \quad \frac{1}{\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|} = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| \quad \text{when the limit exists}$$

Continuity

- f is continuous on $(x_0 - R, x_0 + R)$ and any $[a,b] \subseteq (x_0 - R, x_0 + R)$

Term by term differentiation

- $f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}$ for $x \in (x_0 - R, x_0 + R)$
- $f \in C^{\infty}(x_0 - R, x_0 + R)$, $\quad f^{(n)}(x_0) = n!\, a_n$

Abel's Thm

- If $f(x)$ converges at $x_0 - R / x_0 + R \Rightarrow f$ converges uniformly on $[x_0 - R, x_0] / [x_0, x_0 + R]$ and f continuous at the endpoint $x_0 - R / x_0 + R$

Term by term Integration

- If f converges on $[a,b]$, then f is integrable on $[a,b]$ and we have term by term integration

Analytic function

Multiplying Series Thm Let $a_k, b_k \in \mathbb{R}$, $c_k = \sum_{j=0}^{k} a_j b_{k-j}$

- Suppose $\sum a_k$ and $\sum b_k$ converge and at least one of them converge absolutely

  $\Rightarrow \sum c_k = (\sum a_k)(\sum b_k)$ converges
- Suppose $\sum a_k x^k$ and $\sum b_k x^k$ both converge on $(-R, R)$

  $\Rightarrow \sum c_k x^k = (\sum a_k x^k)(\sum b_k x^k)$ converges on $(-R, R)$
- Suppose $\sum a_k$, $\sum b_k$, $\sum c_k$ all converge

  $\Rightarrow \sum c_k = (\sum a_k)(\sum b_k)$

Analytic functions (locally expressible as a power series)

f is analytic on $(a,b) \Rightarrow \forall x_0 \in (a,b)$, $\exists$ open interval $(c,d) \subseteq (a,b)$
$f \in C^\infty(a,b)$ w/ $x_0 \in (c,d)$ s.t.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \quad \forall x \in (c,d)$$

Analytic M-test

Let $f \in C^\infty(a,b)$, if $\exists M > 0$ s.t. $|f^{(n)}(x)| \leq M^n$, $\forall x \in (a,b)$, $\forall n \in \mathbb{N}$

$\Rightarrow f$ is analytic on (a,b) and $\forall x_0 \in (a,b)$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \qquad \forall x \in (a,b)$$

A power series is Analytic

A power series is analytic on its interval of convergence $(-R, R)$, and $\forall x_0 \in (-R, R)$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \quad \text{for } x \text{ w/ } |x-x_0| < R - |x_0|$$

Identity Thm

If f and g are analytic on (a,b) and $f=g$ on $(c,d) \subseteq (a,b)$

$\Rightarrow f = g$ on (a,b)