# Series Convergence

## Convergence of series

Let $S_n := \sum_{k=1}^{n} a_k$, $\sum_{k=1}^{\infty} a_k$ converges if $\{S_n\}$ converges

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$$

## Divergence test

If $a_k \not\to 0$, then $\sum_{k=1}^{\infty} a_k$ diverges

## Geometric series test

$$\sum_{k=0}^{\infty} ar^k \text{ converges} \iff |r| < 1$$

and in this case $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

## Cauchy Criterion

$\sum_{k=1}^{\infty} a_k$ converges $\iff \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon \quad \forall m \geq n \geq N$$

## Tails converge to 0

$\sum_{k=1}^{\infty} a_k$ converges $\Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon \quad \forall n \geq N$$

or $\sum_{k=n+1}^{\infty} a_k \to 0$ as $n \to \infty$

## Convergence w/ nonnegative terms

$a_k \geq 0 \quad \forall k \in \mathbb{N}$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \{S_n\} \text{ converges}$$

$$\iff \{S_n\} \text{ is bounded above}$$

## p-series test

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \iff p > 1$$

## Comparison test

Let $0 \leq a_k \leq b_k$ for large $k$

$$\sum_{k=1}^{\infty} b_k \text{ conv} \implies \sum_{k=1}^{\infty} a_k \text{ conv}$$

$$\sum_{k=1}^{\infty} a_k \text{ div} \implies \sum_{k=1}^{\infty} b_k \text{ div}$$

## Limit Comparison test

$a_k \geq 0$, $b_k > 0$ for large $k$, $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = L$

i) $L > 0$ $\qquad \displaystyle\sum_{k=1}^{\infty} a_k \text{ conv} \iff \sum_{k=1}^{\infty} b_k \text{ conv}$

ii) $L = 0$ $\qquad \displaystyle\sum_{k=1}^{\infty} b_k \text{ conv} \implies \sum_{k=1}^{\infty} a_k \text{ conv}$

$$\sum_{k=1}^{\infty} a_k \text{ div} \implies \sum_{k=1}^{\infty} b_k \text{ div}$$

Note: $\sqrt[n]{n} \to 1$ as $n \to \infty$

## Abel's formula

$\forall m \geq n \geq 1 \qquad \displaystyle\sum_{k=n}^{m} a_k b_k = A_m b_m - A_{n-1} b_n + \sum_{k=n}^{m-1} A_k (b_k - b_{k+1})$

## Dirichlet's test

If $\{A_n\}$ is bounded and $b_k \downarrow 0$ as $k \to \infty$

then $\sum_{k=1}^{\infty} a_k b_k$ converges

## Alternating series test

If $b_k \downarrow 0$ as $k \to \infty$

then $\sum_{k=1}^{\infty} (-1)^{k+1} b_k$ converges

# Absolute Convergence

## Absolute Convergence

$\sum_{k=1}^{\infty} a_k$ converges absolutely $\iff$ $\sum_{k=1}^{\infty} |a_k|$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges

$\sum_{k=1}^{\infty} a_k$ converges conditionally $\iff$ $\sum_{k=1}^{\infty} |a_k|$ diverges and $\sum_{k=1}^{\infty} a_k$ converges

## Rearrangement

$\sum_{j=1}^{\infty} b_j$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$ if

$\exists$ a bijection $f: \mathbb{N} \to \mathbb{N}$ s.t.

$a_k = b_{f(k)} \quad \forall k \in \mathbb{N}$

## Rearrangement Thm

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, and $\sum_{j=1}^{\infty} b_j$ is any rearrangement of $\sum_{k=1}^{\infty} a_k$

then $\sum_{j=1}^{\infty} b_j = \sum_{k=1}^{\infty} a_k$ so it converges

## Lim Sup

Let $\{x_n\}$ be a seq in $\mathbb{R}$, $x \in \mathbb{R}$

i) $\limsup x_n < x \Rightarrow x_n < x$ for large $n$

ii) $\limsup x_n > x \Rightarrow x_n > x$ for infinitely many $n$

iii) $x_n \to x \Rightarrow \limsup x_n = x$

iv) $x_n \to \infty \Rightarrow \limsup x_n = \infty$

v) $x_n \to -\infty \Rightarrow \limsup x_n = -\infty$

## Root test

let $r := \limsup_{k\to\infty} |a_k|^{1/k} \in \mathbb{R} \cup \{\infty\}$

i) $r<1 \Rightarrow \sum_{k=1}^{\infty} a_k$ conv abs

ii) $r>1 \Rightarrow \sum_{k=1}^{\infty} a_k$ div

iii) $r=1 \Rightarrow$ inconclusive

## Ratio test

let $a_k \neq 0$ for large $k$, $r := \limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$

i) $r<1 \Rightarrow \sum_{k=1}^{\infty} a_k$ conv abs

ii) $\left|\frac{a_{k+1}}{a_k}\right| \geq 1$ for large $k$, then $\sum_{k=1}^{\infty} a_k$ div

Rmk

i) $\lim \left|\frac{a_{k+1}}{a_k}\right| < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ conv abs

ii) $\lim \left|\frac{a_{k+1}}{a_k}\right| > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ div

iii) $\lim \left|\frac{a_{k+1}}{a_k}\right| = 1 \Rightarrow$ inconclusive

# Uniform Convergence

## Pointwise Convergence

Let $\emptyset \neq E \subseteq \mathbb{R}$, $f_n$, $f: E \to \mathbb{R}$, $n \in \mathbb{N}$

Say $f_n \to f$ pointwise on $E$ and call $f$ the pointwise limit of $f_n$

if: $\forall x \in E$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$|f_n(x) - f(x)| < \varepsilon$$

## Uniform Convergence

Say $f_n \to f$ uniformly on $E$ and call $f$ the uniform limit of $\{f_n\}$
if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$:

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall x \in E$$

## Uniform Convergence Lemma

If $\exists a_n \in \mathbb{R}$ s.t.

$$|f_n(x) - f(x)| \leq a_n \quad \forall x \in E$$

and $a_n \to 0$, then $f_n \to f$ uniformly on $E$

## Uniform Convergence Preserves boundedness

If $f_n \to f$ uniformly on $E$, each $f_n$ bounded on $E$,
then $f$ is bounded on $E$

## Reverse Triangle Inequality

$$\big||x| - |y|\big| \leq |x - y| \Rightarrow |x| + |y| \leq |x - y|$$

## Uniform Convergence preserves Continuity (and uniform Continuity)

Let $\emptyset \neq E \subseteq \mathbb{R}$, $a \in E$, If $f_n \to f$ uniformly on $E$, and each $f_n$ is continuous at $a$, then $f$ is continuous at $a$

## Uniform Cauchy Criterion

$\{f_n\}$ converge uniformly on $E$ $\iff$ $\{f_n\}$ is uniformly Cauchy on $E$

$\iff$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \geq N \quad \forall x \in E$$

## Uniform Convergence preserves Integration

Let $a < b$ in $\mathbb{R}$, if $f_n \to f$ uniformly on $[a,b]$ and each $f_n$ is integrable, then $f$ is integrable on $[a,b]$ and

$$\int_a^b f_n \to \int_a^b f \qquad \left(\text{i.e. } \lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n\right)$$

moreover, $\int_a^x f_n \to \int_a^x f$ uniformly for $x \in [a,b]$