Metric Spaces

Metric Definition

Let X be a set, $\rho: X \times X \to \mathbb{R}$ is a metric such that

- positive-definite: $\rho(x,y) \geq 0$, $\rho(x,y) = 0 \iff x = y$
- symmetric: $\rho(x,y) = \rho(y,x)$
- Δ-inequality: $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$

Open & Closed balls

Let $a \in X$, $r > 0$

Open ball: $x \in B_r(a) \iff x \in X$ and $\rho(x,a) < r$

Closed ball: $x \in B_{\leq r}(a) \iff x \in X$ and $\rho(x,a) \leq r$

Open & closed metric spaces

Let $V, E \subseteq X$

- V is open in $X \iff \forall x \in V$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq V$
- E is closed in $X \iff E^c$ is open in X

(Sequential characterization of closed sets) $\iff \forall x_k \to x$ w/ each $x_k \in E$ implies $x \in E$

Completeness

A metric space (X,ρ) is complete if every Cauchy sequence converges in (X,ρ)

Sequentially compact

Let $E \subseteq X$

E is sequentially compact $\Rightarrow$ Every bounded sequence in E has a subsequence converging in E

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Limit of functions

Cluster point Let (X, ρ) be a metric space, $E \subseteq X$, $a \in X$

a is a cluster point of E $\iff$ $\forall \delta > 0$, $B_\delta(a) \cap E$ contains infinitely many points

$\iff$ $\forall \delta > 0$, $B_\delta(a) \cap E \setminus \{a\} \neq \emptyset$

$\iff$ $\exists$ a sequence $\{x_n\}$ in $E \setminus \{a\}$ s.t. $x_n \to a$

Continuity: $E \subseteq X$

$f: E \to Y$ is continuous at $a \in E$ if

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\tau(f(x), f(a)) < \varepsilon$ $\forall x \in E$ w/ $\rho(x, a) < \delta$

SCL

$\lim_{x \to a} f(x) = L$ $\iff$ $\forall$ sequences x_n w/ each $x_n \in E$ s.t. $x_n \to a$, $f(x_n) \to L$

SCC

f is continuous at $a \in E$ $\iff$ $\forall$ sequences x_n w/ each $x_n \in E$ s.t. $x_n \to a$, $a \in E$, then $f(x_n) \to f(a)$

Continuous functions:

- the coordinate function $\mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto x_j$ is continuous
- polynomials are continuous on $\mathbb{R}^n$
- every rational of polynomials is continuous on $\mathbb{R}^n$
- composite of continuous functions are continuous

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Interior, Closure, Boundary

Set unions and intersections Let (X, ρ) be a metric space, A be an index set

- each V_α is open in X, $\alpha \in A \Rightarrow \bigcup_{\alpha \in A} V_\alpha$ is open
- each V_k is open in X, $k=1,\dots,n \Rightarrow \bigcap_{k=1}^{n} V_k$ is open
- each E_α is closed in X, $\alpha \in A \Rightarrow \bigcap_{\alpha \in A} E_\alpha$ is closed
- each E_k is closed in X, $k=1,\dots,n \Rightarrow \bigcup_{k=1}^{n} E_k$ is closed

Interior, Closure, Boundary Let $E \subseteq X$, $x \in X$

- Interior: $x \in E^\circ \iff \exists r>0$ s.t. $B_r(x) \subseteq E$
  $\iff x$ is an interior point of E
- Closure: $x \in \bar{E} \iff \forall r>0$, $B_r(x) \cap E \neq \emptyset$
  $\iff x$ is a limit point of E
- Boundary: $x \in \partial E \iff \forall r>0$, $B_r(x) \cap E \neq \emptyset$ and $B_r(x) \cap E^c \neq \emptyset$
  $\iff x$ is a boundary point of E

- $E^\circ \subseteq E \subseteq \bar{E}$
- E is open $\iff E = E^\circ$
- E is closed $\iff E = \bar{E}$
- ∂E is closed

Interiors Let $E, A, B \subseteq X$

- $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$
- E° is open
- If V is open and $V \subseteq E$, $V \subseteq E^\circ$ (E° is the largest open set contained in E)
- $E^\circ = \bigcup\{V : V \text{ is open and } V \subseteq E\}$
- $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$, $(A \cap B)^\circ = A^\circ \cap B^\circ$

Closure Let $E, A, B \subseteq X$

- $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$
- $\bar{E}$ is closed
- If F is closed and $E \subseteq F$, then $\bar{E} \subseteq F$ ($\bar{E}$ is the smallest closed set containing E)
- $\bar{E} = \bigcap\{F : F \text{ is closed and } E \subseteq F\}$
- $\overline{A \cup B} = \bar{A} \cup \bar{B}$, $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Duality and Boundary Let $E \subseteq X$

- $(\bar{E})^c = (E^c)^\circ$, $(E^\circ)^c = \overline{E^c}$
- $\partial E = \bar{E} \cap \overline{E^c} = \bar{E} \setminus E^\circ$

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