• Lim Sup and Lim inf:

Let x_n be bounded, then

- $\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k = \inf_{n\in\mathbb{N}} \sup_{k\geq n} x_k$ exists
- $\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \inf_{k\geq n} x_k = \sup_{n\in\mathbb{N}} \inf_{k\geq n} x_k$ ★

• Limits of Functions:

$\lim_{x\to a} f(x) = L$ or $f(a) \to L$ as $x \to a$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ $\forall x \in I$ w/ $0 < |x-a| < \delta$

• Limits of sequences: ★ From Midterm 1

$\lim_{n\to\infty} x_n = a$ or $x_n \to a$

if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - a| < \varepsilon$

• Sequential characterization of Limits

$f(x) \to L$ as $x \to a$ $\iff$

$f(x_n) \to L$ ∀ sequences $x_n \to a$ w/ each $x_n \in I \setminus \{a\}$

• Divergence Criteria

If ∃ sequence $x_n \to a$ w/ $x_n \in I \setminus \{a\}$ s.t. $\{f(x_n)\}$ diverges,

or ∃ sequences $x_n \to a$ and $y_n \to a$ w/ $x_n, y_n \in I \setminus \{a\}$ s.t. $\{f(x_n)\}$ and $\{f(y_n)\}$ converges to different limits,

then $\lim_{x\to a} f(x)$ DNE

Limits of functions

Note: $\{\sin(\frac{\pi}{2} + n\pi)\} = \{(-1)^n\}$ $\forall n \in \mathbb{N}$

- Algebraic Laws of limits of functions:

  Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

  Then $\lim_{x \to a} (f+g)(x) = L + M$, $\lim_{x \to a} (\alpha f)(x) = \alpha L$ $\forall \alpha \in \mathbb{R}$

  $\lim_{x \to a} (fg)(x) = LM$, $\lim_{x \to a} \frac{f}{g}(x) = \frac{L}{M}$ if $M \neq 0$

- Squeeze Thm:

  If $f(x) \leq g(x) \leq h(x)$, $\forall x \in I \setminus \{a\}$, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

- Comparison Thm:

  If $f(x) \leq g(x)$, $\forall x \in I \setminus \{a\}$, $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$,

  then $L \leq M$

Let $f : (a,b) \to \mathbb{R}$, where $a < b$

- Right hand limit:

  $\lim_{x \to a^+} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

  $|f(x) - L| < \varepsilon$ $\forall x \in (a,b)$ w/ $a < x < a + \delta$

- Left hand limit:

  $\lim_{x \to b^-} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

  $|f(x) - L| < \varepsilon$ $\forall x \in (a,b)$ w/ $b - \delta < x < b$

Continuity

Let $f: E \to \mathbb{R}$, where $\emptyset \neq E \subseteq \mathbb{R}$

- **Continuous function:**

  f is continuous at $a \in E$ if $\forall \varepsilon > 0$, $\exists \delta > 0$
  s.t. $|f(x) - f(a)| < \varepsilon$ $\forall x \in E$ w/ $|x-a| < \delta$

  say f is continuous on E if f is continuous at every $a \in E$

- **Sequential Characterization of Continuity**

  let $f: E \to \mathbb{R}$, where $\emptyset \neq E \subseteq \mathbb{R}$, $a \in E$
  Then f is continuous at a
  $\iff$ $f(x_n) \to f(a)$ $\forall$ sequences $x_n \to a$ w/ each $x_n \in E$

- **Algebraic Laws of continuous functions**

  Let $f, g: E \to \mathbb{R}$, where $\emptyset \neq E \subseteq \mathbb{R}$,

  $a \in E$, if f and g are continuous at a, then so are
  - $f+g$,
  - αf, $\forall \alpha \in \mathbb{R}$,
  - fg,
  - $\frac{f}{g}$ if $g(a) \neq 0$, $\frac{1}{g}$

  Every polynomial is continuous, and $\frac{P(x)}{Q(x)}$ (P, Q are polynomials)

- **Composite functions**

  Let $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$, where $A, B \subseteq \mathbb{R}$, $f(A) \subseteq B$

  $(g \circ f)(x) = g(f(x))$, $x \in A$
  if f is continuous at $a \in A$ and g is continuous at $f(a)$,
  then $g \circ f$ is continuous at a

- Let $f,g: E \to \mathbb{R}$, if f and g are continuous on E, so are:
  - $|f|$, $|g|$
  - $|f+g|$, $|f-g|$
  - $\max\{f(x), g(x)\}$

- Extreme Value Thm ★ ☆

  Let $f:[a,b] \to \mathbb{R}$ be continuous, where $a \le b$ in $\mathbb{R}$, then
  i) f is bounded i.e. $\exists M \in \mathbb{R}$ s.t. $|f(x)| \le M$ $\forall x \in [a,b]$
  ii) f attains a minimum and a maximum

- Approximation property for a sequence converging to supremum

  Let S be bounded and $\emptyset \neq S \subseteq \mathbb{R}$,
  Let $L = \sup(S)$, By the Approximation property,
  $\forall \varepsilon > 0$, $\exists a \in S$ s.t. $L - \varepsilon < a \le L$
  let $\varepsilon = \frac{1}{n}$, we have $\{x_n\} \subseteq S$ s.t.

  $$L - \frac{1}{n} < x_n \le L, \text{ by squeeze, } \lim_{n \to \infty} x_n = L$$

- Intermediate Value Thm: ★ ☆

  Let $f:[a,b] \to \mathbb{R}$ be continuous, where $a < b$ in $\mathbb{R}$
  If $f(a) < y_0 < f(b)$, $\exists x_0 \in (a,b)$ s.t. $f(x_0) = y_0$

- Trig Ineq:

  $$|a+b| \le |a| + |b|$$
  $$\big||a| - |b|\big| \le |a-b|$$

Uniform Continuity

- **Uniformly continuous function:**

  Let $f: E \to \mathbb{R}$, where $\emptyset \neq E \subseteq \mathbb{R}$

  f is uniformly continuous on E if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ $\forall x, y \in E$ w/ $|x-y| < \delta$

  ★ δ only depend on ε

- **Uniform continuity and Cauchy**

  Let $f: E \to \mathbb{R}$ be uniformly continuous, where $\emptyset \neq E \subseteq \mathbb{R}$, if $\{x_n\}$ is cauchy w/ $x_n \in E$, then $\{f(x_n)\}$ is Cauchy

- **Continuous to uniformly continuous Thm**

  Any continuous function $f: [a,b] \to \mathbb{R}$ is uniformly continuous, where $a \leq b$

Differentiability

- **Differentiable function**

  Let $a \in \mathbb{R}$, I an open interval containing a, $f: I \to \mathbb{R}$, say f is differentiable at a if

  $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  exists, and f is differentiable on I if f is differentiable on every $a \in I$. $f''(a) = (f')'(a)$, in general, $f^{(n)}(a) = (f^{(n-1)})'(a)$, $n \in \mathbb{N}$

- **Continuously differentiable**

  f is continuously differentiable on I and write $f \in C'(I)$ if f' exists and is continuous on I

$$C^n(I) = \left\{ f: I \to \mathbb{R},\ f \text{ is } n \text{ times continuously differentiable} \right.$$

$n \in \mathbb{N}$

$$C^\infty(I) = \left\{ f: I \to \mathbb{R}: f \text{ is smooth or indefinitely differentiable},\ f^{(n)} \text{ exists on } I\ \forall n \in \mathbb{N} \right.$$

- **Differentiability on non-open intervals**

$f: [a,b] \to \mathbb{R}$,

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$f'(b) = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

- **Differentiability Implies Continuity Thm**

If f is differentiable at a, then f is continuous at a

- **Chain Rule**

Let $f: I \to \mathbb{R}$, $g: J \to \mathbb{R}$, where I, J are open intervals, and $f(I) \subseteq J$

Suppose f is diff'ble at $a \in I$ and g is diff'ble at $f(a)$,
Then $g \circ f$ is diff'ble at a, and $(g \circ f)'(a) = g'(f(a)) f'(a)$

- **Algebraic Rules of differentiable functions**

Let $f, g: I \to \mathbb{R}$ be differentiable at a, where I is an interval containing a,
Then $f+g$, αf ($\forall \alpha \in \mathbb{R}$), fg, $\frac{f}{g}$ ($g(a) \neq 0$) are all differentiable at a

- $(f+g)'(a) = f'(a) + g'(a)$
- $(\alpha f)'(a) = \alpha f'(a)$
- $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
- $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$, $\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{g^2(a)}$

- **Power Rule**

$(x^n)' = nx^{n-1}$

Mean Value Theorem

• Rolle's Thm:

Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b)

If $f(a) = f(b)$, then $\exists\, c \in (a,b)$
s.t. $f'(c) = 0$

• Mean Value Thm:

Let $f, g: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$
and differentiable on (a,b),
then $\exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

more generally, $\exists\, c \in (a,b)$ s.t.

$$f'(c) = (g(b) - g(a)) = g'(c)(f(b) - f(a))$$

• strictly or non strictly increasing or decreasing

Let $f: [a,b] \to \mathbb{R}$ be cts on $[a,b]$
and diff'ble on (a,b)

f is:

• strictly increasing if $f'(x) > 0$	$\forall x \in (a,b)$	$f(x) < f(y)$	if $x < y$
• increasing if $f'(x) \geq 0$	$\downarrow$	$f(x) \leq f(y)$	$\downarrow$
• strictly decreasing if $f'(x) < 0$		$f(x) > f(y)$	
• decreasing if $f'(x) \leq 0$		$f(x) \geq f(y)$	
• constant if $f'(x) = 0$		$f(x) = f(y)$	

• Intermediate Value Thm for derivatives

Let $f: [a,b] \to \mathbb{R}$ be diff'ble w/ $f'(a) \neq f'(b)$,
If $f'(a) < y_0 < f'(b)$ or $f'(b) < y_0 < f'(a)$,
then $\exists\, c \in (a,b)$ s.t. $f'(c) = y_0$

- **Heine Cantor Thm**

  If $f:[a,b] \to \mathbb{R}$ is differentiable w/ $f'(x)$ being bounded on (a,b), then f is uniformly continuous

- **Second derivative test**

  Let $a \in \mathbb{R}$, I an open interval containing a, $f: I \to \mathbb{R}$ differentiable.
  If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a, i.e. $\exists\, \delta > 0$ w/ $(a-\delta, a+\delta) \subseteq I$ s.t. $f(x) \geq f(a)$ $\forall\, x \in (a-\delta, a+\delta)$

- Cauchy Sequences:

  $\{x_n\}$ is Cauchy if $\forall\, \varepsilon > 0$,
  $\exists\, N \in \mathbb{N}$ s.t. $|x_m - x_n| < \varepsilon$, $\forall\, n, m \geq N$