Taylor Polynomial

Taylor Polynomial: ✡ Continuous functions are approximated by polynomials

Let f be n times differentiable at a,
define nth polynomial of f centered at a to be:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

and the nth Taylor Remainder to be:

$$R_n(x) = f(x) - P_n(x)$$

Taylor's Thm: ✡ Express $R_n(x)$ in terms of $f^{(n+1)}$

Let $f:(a,b) \to \mathbb{R}$ be nth time differentiable,
where $-\infty \le a < b \le \infty$, $n \in \mathbb{N} \cup \{0\}$
then $\forall x, x_0 \in (a,b)$, $\exists\, c$ between x and x_0 s.t.

$$f(x) = \underbrace{f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{nth Taylor polynomial of } f \text{ centered at } x_0} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{\text{nth taylor polynomial remainder}}$$

Partitions

Partition definition

i) A partition of $[a,b]$ is a finite set of points
$P = \{x_0, x_1, x_2, \dots, x_n\}$ s.t. $a = x_0 < x_1 < x_2 < \dots < x_n = b$

ii) Let $f:[a,b] \to \mathbb{R}$ be bounded,
$P = \{x_0, x_1, \dots, x_n\}$ a partition of $[a,b]$

- define the upper sum of f w.r.t. P to be

$$U(f,P) = \sum_{k=1}^{n} \sup_{x \in [x_{k-1}, x_k]} f(x) \cdot (x_k - x_{k-1})$$

- define the lower sum to be

$$L(f,P) = \sum_{k=1}^{n} \inf_{x \in [x_{k-1}, x_k]} f(x) \cdot (x_k - x_{k-1})$$

Note: $L(f,P) \le U(f,P)$

Let P, Q be partitions of $[a,b]$, if Q is a refinement of P ($P \subseteq Q$, P contains at least one more point)

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

Integration

Notation: Let $f:[a,b] \to \mathbb{R}$ be unbounded, where $a<b$ in $\mathbb{R}$
Let P be the collection of all partitions of $[a,b]$

Cor:

$$L(f,S) \le U(f,Q) \quad \forall S, Q \in P$$

$$\sup_{S\in P} L(f,S) \le \inf_{Q\in P} U(f,Q)$$

Lower and upper integral

i) The Upper integral of f on $[a,b]$ is

$$\overline{\int_a^b} f(x)\,dx := \inf_{P\in P} U(f,P)$$

ii) The lower integral of f on $[a,b]$ is

$$\underline{\int_a^b} f(x)\,dx := \sup_{P\in P} L(f,P)$$

Note: $\underline{\int_a^b} f \le \overline{\int_a^b} f$

iii) Say the function f is integrable on $[a,b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f := \int_a^b f$$

Comparison Thm of integrals

Let f,g be integrable on $[a,b]$, where $a<b$ in $\mathbb{R}$
If $f(x) \le g(x)$, $\forall x \in [a,b]$, then $\int_a^b f \le \int_a^b g$
in particular, if $m \le f(x) \le M$, $\forall x \in [a,b]$
then $m(b-a) = \int_a^b m \le \int_a^b f \le \int_a^b M = M(b-a)$

Approximation Characterization of Integrability

Let $f:[a,b]\to\mathbb{R}$ be bounded, with $a<b$ in $\mathbb{R}$, and P be the set of all possible partitions of $[a,b]$

$$f \text{ is integrable on } [a,b] \iff \forall \varepsilon>0,\ \exists P_\varepsilon \in P \text{ s.t. } U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$$

Integrability of continuous functions

Let $f:[a,b]\to\mathbb{R}$ be continuous, where $a<b$ in $\mathbb{R}$, then f is integrable on $[a,b]$

Domain splitting

Let $f:[a,b]\to\mathbb{R}$ be bounded, where $a<b$, $c\in[a,b]$

then:

f is integrable on $[a,b]$ $\Longleftrightarrow$ f is integrable on $[a,c]$ and $[c,b]$

Domain restriction

if f is integrable on $[a,b]$, then it is integrable on $[c,d]$ $\forall c<d$ in $[a,b]$

Endpoints don't matter

Let $f:[a,b]\to\mathbb{R}$ be integrable, where $a<b$ in $\mathbb{R}$, If $g:[a,b]\to\mathbb{R}$ satisfies $g(x)=f(x)$ $\forall x\in(a,b)$, then g is integrable on $[a,b]$, and

$$\int_a^b f=\int_a^b g$$

Finitely many points don't matter

i) Let $f:[a,b]\to\mathbb{R}$ be integrable, where $a<b$ in $\mathbb{R}$, If $g:[a,b]\to\mathbb{R}$ satisfies $g(x)=f(x)$ for all but finitely many $x\in[a,b]$, then g is integrable on $[a,b]$, and

$$\int_a^b f=\int_a^b g$$

ii) Let $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$ and continuous at all but finitely many points of $[a,b]$, then f is integrable on $[a,b]$

FTC

Fundamental Theorem of Calculus

$f: [a,b] \to \mathbb{R}$

i) If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$

$\Rightarrow F \in C^1[a,b]$, $\frac{d}{dx}\int_a^x f(t)\,dt = F'(x) = f(x)$ for each $x \in [a,b]$

ii) If f is differentiable on $[a,b]$ and f' is integrable on $[a,b]$

$\Rightarrow \int_a^x f'(t)\,dt = f(x) - f(a)$ for each $x \in [a,b]$