MATH 327: Classical Mechanics Notes

Newtonian, Lagrangian, and Hamiltonian Mechanics

Newtonian Mechanics

- **Newton's Law**

$$F(t, x(t), \dot{x}(t)) = \ddot{x}(t).$$

- **Picard-Lindelof Thm**

$$\dot{x} = G(t, x), \quad x(0) = x_0.$$

If G is differentiable, then $\forall x_0, V_0,\ \exists T > 0$ and a unique solution $x(t)$

for $-T < t < T,\ x(0) = x_0,\ \dot{x}(0) = v_0$.

- **Method using Taylor series**

For equation G, we may write

$$x(t) = \sum_{j=0}^{\infty} \frac{d^j x}{dt^j}(0)\frac{t^j}{j!}.$$

- **Method from Euler-Rom scheme**

Let $x(t+h) = x(t) + \dot{x}(t)h, \qquad x(t+2h) = x(t+h) + \dot{x}(t+h)h.$

- **Method from Picard iteration**

Let $x_0(t) = x(0), \qquad x_{j+1}(t) = x_0(0) + \int_0^t G(s, x_j(s))\, ds.$

- **Conservation of Energy**

$$F = -\nabla V(x).$$

Given $F(x)$, 1) calculate $V(x) = -\int F(x)\, dx, \qquad E = \frac{1}{2}\dot{x}^2 + V(x).$

2) $\dot{x} = \sqrt{2(E - V(x))}.$

3) integrate to solve: $x(t) = x^{-1}(t + c).$

- **Equilibrium Points**

$$\text{An equilibrium point } (x_0, 0) \in \mathbb{R}^2 \text{ satisfies } F(x_0) = 0 \iff V'(x_0) = 0.$$
$$\text{It is stable if } V''(x_0) > 0, \qquad \text{unstable if } V''(x_0) < 0.$$

- **Phase portraits**

Given a potential $V(x)$, we can plot the phase portrait by
finding all equilibrium points and determining their stability.

- **Special case of Poincare Lemma**

$$\nabla \times F = 0, \text{ i.e. } \partial_i F_j = \partial_j F_i, \quad \forall i, j \iff \exists \text{ smooth } V \in C^1 \text{ s.t. } F = -\nabla V.$$
$$\text{If this holds, } F \text{ is conservative, and } E = \frac{1}{2}\|\dot{x}\|^2 + V(x) \text{ is conserved.}$$

- **Bound and unbound motion**

$$\text{A trajectory } x : \mathbb{R} \to \mathbb{R}^n \text{ is bound if } \exists R > 0 \text{ s.t. } \|x(t)\| < R \ \forall t.$$
Otherwise it is unbound.

- **Bound motion proposition**

$$\text{Suppose } F : \mathbb{R}^n \to \mathbb{R}^n \text{ conservative, with potential } V.$$
$$\text{If } \exists R > 0 \text{ s.t. } \|x\| \leq R \text{ for all } \|x\| > R, \text{ then if } E < 0, \text{ the motion is bound.}$$

Lagrangian Mechanics

- **Lagrangian**

The Lagrangian $L = T - V$,
where T is the kinetic energy of the system and V is the potential energy.

- **Stationary action**

γ is a stationary point of the action $\Longleftrightarrow$ Euler-Lagrange equation holds:

$$L_x(t, \gamma, \dot{\gamma}) = \frac{d}{dt} L_v(t, \gamma, \dot{\gamma}).$$

- **Noether's Thm**

Let $\gamma_T : x \mapsto x + w(x) + O(T^2)$ be a continuous one parameter family,

$$\text{where } w(x) = \frac{d}{dT}\gamma_T(x)\Big|_{T=0} \quad \text{s.t.} \quad L(Tx, D\gamma_T(x), v) = L(x, v).$$

Then $p \cdot w = \dfrac{\partial L}{\partial v} \cdot w$ is conserved.

- **Conservation of Hamiltonian**

If L does not depend on time, then the Hamiltonian H is conserved.

Hamiltonian Mechanics

- **Legendre transform**

$$\text{Given Lagrangian } L, \quad p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad \dot{q}_i = x_i \text{ where map } x(q,p).$$

$$\text{Also require } L_{ij} \neq 0 \Rightarrow p_i = 0.$$

- **Hamiltonian**

$$\text{The Hamiltonian } H(q,p) = \sum_i p_i \dot{q}_i - L(q, \dot{q}).$$

- **Hamilton's equations**

$$\text{If } x(t) \text{ solves E-L, then } q(x(t)), p(x(t)) \text{ satisfy} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

- **Hamilton flow**

Hamilton H generates a flow in phase space $\mathbb{R}^{2n}$,

$$X_H = \begin{pmatrix} \partial H / \partial p \\ -\partial H / \partial q \end{pmatrix},$$

where (q,p) evolves along an integral curve of the vector field.

- **Liouville's Thm**

Let $\phi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the Hamiltonian flow.

$$\int_{\phi_t(U)} 1 \, dV = \int_U 1 \, dV, \qquad \text{i.e. } \phi \text{ is volume preserving.}$$

- **Application in Stat Mech**

$$\rho : \mathbb{R}^{2n} \to \mathbb{R} \text{ is a density if } (i) \; \rho \geq 0 \text{ everywhere}, \quad (ii) \int_{\mathbb{R}^{2n}} \rho(q,p) \, dq \, dp = 1.$$

$\rho \circ \phi_t$ is a density.

The flow generated by a vector field V is volume preserving $\iff \nabla \cdot V = 0$.

- **Poisson brackets**

Let $a(q,p) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be an observable.

$$\{q_i, H\} = \dot{q}_i, \qquad \{p_i, H\} = \dot{p}_i, \qquad \{f, g\} = -\{g, f\}.$$

$$\{f, g\}h + g\{f, h\} + \{h, f\}g = 0.$$

$$\text{Can: if } \{H, H\} = 0,\ \{f, H\} = 0,\ \text{then } \{f + g, H\} = 0.$$

Formula page

- Tips paper
- Spherical coordinates
- Symmetry group