

# MATH 325: Complex Analysis Notes

Midterm and Final Review

## Complex Numbers and Functions

- Complex number

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}, \quad |z|^2 = z\bar{z}.$$

Euler's formula:  $e^{i\theta} = \cos \theta + i \sin \theta$  ( $\theta \in \mathbb{R}$ ).

$$i^n = \begin{cases} 1, & n \equiv 0 \pmod{4}, \\ i, & n \equiv 1 \pmod{4}, \\ -1, & n \equiv 2 \pmod{4}, \\ -i, & n \equiv 3 \pmod{4}. \end{cases}$$

- De Moivre's formula

$$\forall n \in \mathbb{N}, \quad (\cos \theta + i \sin \theta)^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta).$$

- Roots of unity

For the  $n$ th roots of unity  $z^n = 1$ ,  $z_k \in \mathbb{C}$ ,

$$z_k = \exp\left(\frac{2\pi i k}{n}\right), \quad k = 0, 1, \dots, n-1.$$

- Complex exponential

$$\exp : \mathbb{C} \rightarrow \mathbb{C}, \quad e^z = e^x(\cos y + i \sin y).$$

- Complex logarithm

$$\log z = \ln |z| + i \arg z.$$

- Complex powers

$$z^\alpha = e^{\alpha \log z} = e^{\alpha(\ln |z| + i \arg z)}.$$

$a \in \mathbb{R}$ ,  $a \neq 0$ ,  $n \in \mathbb{Z}$ , then  $z^a$  takes a distinct values.

$a \in \mathbb{C}$  and  $a \notin \mathbb{Z}$ , then  $z^a$  takes infinitely many distinct values.

- Trigonometric and hyperbolic functions**

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

- Images of exp**

Let  $x = x_0$  be constant:  $e^z = e^x(\cos y + i \sin y)$ ,  $y \in \mathbb{R}$ .

The image is a circle w/ radius  $e^{x_0}$ .

Let  $y = y_0$  be constant:  $e^z = e^x(\cos y_0 + i \sin y_0)$ ,  $x \in \mathbb{R}$ .

The image is a ray w/ angle  $y_0$ .

- Images of log**

$\log z$  transforms circles into vertical lines and rays into horizontal lines.

## Complex Differentiation

- Analytic**

$f : U \rightarrow \mathbb{C}$  is analytic on an open  $U \subseteq \mathbb{C}$

if  $f$  is diff<sup>ble</sup>  $\forall u \in U$  and  $f'$  is continuous  $\forall u \in U$ .

- Goursat's Thm**

If  $f$  is diff<sup>ble</sup> at every point of  $D$ , then  $f$  is continuous on  $D$ .

- Cauchy-Riemann Equations**

$f(z) = u + iv$  on  $D \subseteq \mathbb{C}$ .

$f$  is diff<sup>ble</sup> at  $z \iff$  C-R equations hold at  $z$ :  $u_x = v_y, \quad u_y = -v_x.$

$\iff$  the partials  $u, v$  exist and are continuous on  $\mathbb{C}$ .

- Constancy Thm**

Suppose  $f$  is analytic on domain  $D \subseteq \mathbb{C}$ .

$(f'(z) = 0 \forall z \in D)$  or  $(|f(z)| \in \mathbb{R} \forall z \in D)$  or  $(f, \bar{f} \text{ analytic on } D)$

$\implies f$  is constant on  $D$ .

## Harmonic Functions

- **Harmonic function**

Let  $u : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  s.t.  $u \in C^2$ .

Say  $u$  is harmonic on  $D$  if  $u_{xx} + u_{yy} = 0$  on  $D$ .

- **Harmonic and Analytic Thm**

If  $u + iv$  is analytic on  $D$ , then  $u$  and  $v$  are harmonic on  $D$ .

- **Harmonic Conjugate**

Let  $u : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be harmonic.

The harmonic conjugate  $v : D \rightarrow \mathbb{R}$  is a harmonic function for which  $f = u + iv$  is analytic on  $D$ .

- **Maybe tested: Existence Thm**

Let  $u$  be harmonic on  $D$ . If  $D$  is simply connected,  
then  $\exists$  a harmonic conjugate  $v$  on  $D$ .

- **Polygonally-connected**

$E \subseteq \mathbb{C}$  is polygonally-connected if any two points in  $E$   
can be connected by a sequence of line segments lying inside the set.

- **Simply connected**

A polygonally-connected region  $D$  is simply-connected if  
the region enclosed by any simple closed curve is contained in  $D$ .

- **Green's Thm**

If  $D \subseteq \mathbb{R}^2$  is a bounded region with positively oriented boundary  $\partial D$ ,

$$\oint_{\partial D} P dx + Q dy = \iint_D (Q_x - P_y) dA.$$

# Complex Integration

- Line Integrals

Scalar line integral: 
$$\int_{\gamma} f(x, y) ds = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt.$$

Vector line integral: 
$$\int_{\gamma} \vec{F} \cdot d\vec{r} = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt.$$

Complex line integral: 
$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

- ML estimate

Let  $\gamma$  be piecewise smooth,  $f$  continuous on  $\gamma$ .

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz|.$$

If  $L = \ell(\gamma)$ ,  $|f(z)| \leq M$  on  $\gamma$ , then  $\left| \int_{\gamma} f(z) dz \right| \leq ML.$

- FTC

I: Let  $f$  be continuous on  $D \subseteq \mathbb{C}$ ,  $F'(z) = f(z)$  on  $D$ .

Then for  $\gamma$  w/  $\gamma(a) = A$ ,  $\gamma(b) = B$ , 
$$\int_{\gamma} f(z) dz = \int_A^B f(z) dz = F(B) - F(A).$$

II: Let  $D \subseteq \mathbb{C}$  be simply connected,  $f$  analytic on  $D$ .

Then  $f$  has a primitive  $F$  on  $D$ , 
$$F(z) = \int_{z_0}^z f(w) dw.$$

- Cauchy's Thm

Let  $D$  be a bounded domain with  $\partial D$  a finite number of disjoint contours.

If  $f$  is analytic on  $\partial D \cup D$ , then 
$$\int_{\partial D} f(z) dz = 0.$$

- Morera's Thm

Let  $f$  be continuous on a domain  $D$ .

If 
$$\int_c f(z) dz = 0$$
 for every closed rectangle, contour, triangle in  $D$ ,  
then  $f$  is analytic on  $D$ .

- **Cauchy Integral Formula**

Let  $D$  be bounded w/  $\partial D$  a contour; if  $f$  is analytic on  $\partial D \cup D$ ,

$$\text{then } f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz, \quad z_0 \in D.$$

$$\text{Generalized formula: } f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - z_0)^{m+1}} dz, \quad z_0 \in D, m \geq 0.$$

- **Cauchy's Estimates**

Suppose  $f$  is analytic for  $|z - z_0| \leq \rho$ .

If  $\exists M > 0$  s.t.  $|f(z)| \leq M \forall z$  w/  $|z - z_0| \leq \rho$ ,

$$\text{then } \left| f^{(m)}(z_0) \right| \leq \frac{m!}{\rho^m} M.$$

- **Liouville's Thm**

If  $f(z)$  is entire and bounded, then  $f(z)$  is constant.

## Complex Series

- Series convergence

$$\sum_{n=0}^{\infty} a_n \text{ conv} \iff \lim_{N \rightarrow \infty} s_N = \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n \text{ converges.}$$

- Uniform convergence and analyticity

If  $f_n \rightarrow f$  uniformly on  $D$  and each  $f_n$  is analytic on  $D$ ,  
then  $f$  is analytic on  $D$ .

- Taylor's Thm

Suppose  $f(z)$  is analytic for  $|z - z_0| < R$ .

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R, \quad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

$$\text{Power series for } R < \infty : \quad \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = R \quad \text{or} \quad \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \frac{1}{R}.$$

- Uniqueness principle

If  $f, g$  analytic on  $D$  and  $f(z) = g(z)$  for all  $z \in E$ , where  $E \subseteq D$   
contains a nonisolated point, then  $f(z) = g(z) \quad \forall z \in D$ .

- Weierstrass M-test

Suppose  $M_n \geq 0$  and  $\sum M_n$  converges.

If  $|g_n(z)| \leq M_n \quad \forall z \in E, \forall n \in \mathbb{N}$ , then  $\sum_{n=1}^{\infty} g_n(z)$  conv unif and abs on  $E$ .

- Cauchy product formula

$$\left( \sum a_n z^n \right) \left( \sum b_n z^n \right) = \sum c_n z^n, \quad c_n = \sum_{j=0}^n a_j b_{n-j}.$$

## Zeros and Singularities

- Zeros of function**

Suppose  $f$  is analytic and  $f \neq 0$  on  $D$ .

$a$  zero of  $f$ ,  $f(z) = (z - a)^n h(z)$  for  $h(a) \neq 0$ .

$a$  zero has order  $n$  if  $f(a) = \dots = f^{(n-1)}(a) = 0$  and  $f^{(n)}(a) \neq 0$ .

- Laurent series**

If  $f$  is analytic on annulus  $\rho < |z - z_0| < R$ ,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Analytic part:  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ ,      principal part:  $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ .

- Isolated Singularities**

$z_0$  is an isolated singularity of  $f$  if  $f$  is analytic on  $D(z_0, r) \setminus \{z_0\}$ .

Removable:  $\lim_{z \rightarrow z_0} f(z)$  exists.

Pole of order  $N$ :  $f(z) = \frac{\phi(z)}{(z - z_0)^N}$ ,  $\phi$  analytic and  $\phi(z_0) \neq 0$ .

Essential: not removable and not pole; Laurent principal part has infinitely many terms.

- Quotient-pole lemma**

Suppose  $f, g$  analytic in some disk centered at  $z_0$ .

If  $f$  has a zero of order  $m$  at  $z_0$  and  $g$  has a zero of order  $n$  at  $z_0$ ,

$\frac{f(z)}{g(z)}$  has a removable singularity if  $m \geq n$ , and a pole of order  $n - m$  if  $m < n$ .

## Residue

- Residue

Suppose  $f$  has Laurent series expansion  $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$  on  $0 < |z-z_0| < R$ .

$$\text{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \int_{\gamma} f(z) dz.$$

- Residue Thm

Let  $D$  be a domain w/ piecewise smooth boundary  $\partial D$ .

Suppose  $f$  is analytic on  $D \cup \partial D$  except at finite isolated singularities  $z_1, \dots, z_n$ ,

$$\text{then } \int_{\partial D} f(z) dz = 2\pi i \sum_{j=1}^n \text{Res}(f, z_j).$$

- Rules of Residue

Rule 0: If  $f$  has a removable singularity at  $z_0$ , then  $\text{Res}(f, z_0) = 0$ .

Rule 1: If  $f$  has a pole of order 1 at  $z_0$ , then  $\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z-z_0)f(z)$ .

Rule 2: If  $f$  has a pole of order  $N$  at  $z_0$ , then  $\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} ((z-z_0)^N f(z))$ .

Rule 3: If  $f, g$  are both analytic near  $z_0$  and  $g$  has a simple zero at  $z_0$ ,

$$\text{then } \text{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

- Application of Residue

For  $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$  where  $P, Q$  are polynomial and  $\deg Q \geq \deg P + 2$ ,

$$\text{then } \int_{\partial D_R} \frac{P(z)}{Q(z)} dz = \int_{-R}^R \frac{P(x)}{Q(x)} dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz.$$

Evaluate by Residue Thm; arc integral goes to 0 by ML estimate.