MATH 320-3: Real Analysis III Notes

Topology, Differentiability, and Measure

Compactness

- **Covers**

$$\text{Let } E \subseteq X.$$

$$\text{An open cover of } E \text{ is } \mathcal{V} = \{V_\alpha : \alpha \in A\} \text{ of open sets } V_\alpha \text{ in } X$$

$$\text{such that } E \subseteq \bigcup_{\alpha \in A} V_\alpha.$$

$$\mathcal{V} \text{ has a finite subcover if } \exists \text{ a finite set } A_0 \subseteq A \text{ such that } E \subseteq \bigcup_{\alpha \in A_0} V_\alpha.$$

- **Compactness**

$$E \text{ is compact} \iff \text{every open cover of } E \text{ has a finite subcover.}$$

- **Compact iff sequentially compact**

$$\text{In a metric space } X,\ E \subseteq X, \qquad E \text{ is compact} \iff E \text{ is sequentially compact.}$$

- **Heine-Borel**

$$\text{In a metric space } X \text{ with the Bolzano-Weierstrass property, } E \subseteq X,$$

$$E \text{ is compact} \iff E \text{ is closed and bounded.}$$

- **Uniform Continuity**

$$f \text{ continuous on compact } E \implies f \text{ is uniformly continuous on } E.$$

- **Continuity preserves compactness**

$$f \text{ is continuous on compact } E \implies f(E) \text{ is compact in } Y.$$

- **Continuity and pre-image**

$$f \text{ is continuous on } E \iff f^{-1}(U) \text{ is open in } E, \quad \forall U \text{ open in } Y,$$

$$\iff f^{-1}(B) \text{ is closed in } E, \quad \forall B \text{ closed in } Y.$$

- **Extreme Value Theorem**

$$\emptyset \neq E \text{ compact}, \qquad f : E \longrightarrow \mathbb{R} \text{ continuous}$$
$$\Longrightarrow \exists x_{\min}, x_{\max} \in E \text{ such that } f(x_{\min}) = \min_E f, \quad f(x_{\max}) = \max_E f.$$

- **Continuity of inverse function**

$$f \text{ continuous on compact } E,\ f \text{ one-to-one} \Longrightarrow f^{-1} \text{ is continuous.}$$

- **Images and pre-images**

$$\text{Let } X, Y \text{ be sets and } f : X \longrightarrow Y.$$
$$f\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f(E_\alpha), \qquad f\left(\bigcap_{\alpha\in A} E_\alpha\right) \subseteq \bigcap_{\alpha\in A} f(E_\alpha).$$
$$f^{-1}\left(\bigcup_{\alpha\in A} B_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(B_\alpha), \qquad f^{-1}\left(\bigcap_{\alpha\in A} B_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(B_\alpha).$$
$$f(A \cap B) = f(A) \cap f(B) \text{ when } f \text{ is one-to-one.}$$

- **Dense and separable**

$$\text{Let } E \subseteq X.$$
$$E \text{ is dense in } X \Longleftrightarrow \overline{E} = X \Longleftrightarrow \forall x \in X,\ \forall \text{ open } V \ni x,\ V \cap E \neq \emptyset.$$
$$X \text{ is separable if it has a countable dense subset.}$$

- **Lindelof's Theorem**

$$\text{Let } X \text{ be separable and } E \subseteq X.$$
$$\text{Every open cover } \{V_\alpha : \alpha \in A\} \text{ of } E \text{ has a countable subcover.}$$

- **Subspace topology**

$$\text{Let } X \text{ be a metric space, } E \subseteq X,\ U, A \subseteq E.$$
$$U \text{ is open in } E \Longleftrightarrow \exists \text{ an open set } V \text{ in } X \text{ such that } U = V \cap E.$$
$$A \text{ is closed in } E \Longleftrightarrow \exists \text{ a closed set } B \text{ in } X \text{ such that } A = B \cap E.$$

Connectedness

- **Disconnected and connected**

Let (X, ρ) be a metric space, $E \subseteq X$.
E is disconnected if $\exists$ nonempty disjoint sets U, V open in E such that $E = U \cup V$.
E is connected if it is not disconnected.
i.e. if $E = U \cup V,\ U, V$ disjoint and open in $E,$ then $U = \varnothing$ or $V = \varnothing$.

- **Open and closed in connected set**

E is connected $\Longleftrightarrow \varnothing$ and E are the only subsets of E that are both open and closed in E.

- **Intervals are connected**

In $(\mathbb{R}, d)$, E is connected $\Longleftrightarrow E$ is an interval $(a, b), [a, b], (a, b], [a, b)$
or E is a single point.

- **Continuity preserves connectedness**

f continuous on connected $E \Longrightarrow f(E)$ is connected.

- **Intermediate Value Theorem**

$f : E \to \mathbb{R}$ continuous on connected E, $\quad f(x) < y < f(z)$
$\Longrightarrow \exists \xi \in E$ such that $f(\xi) = y$.

- **Path connected**

E is path connected $\Longleftrightarrow \forall x, y \in E,\ \exists$ a continuous function $f : [0, 1] \to E$ such that $f(0) = x,\ f(1) = y$.

- **Path connected implies connected**

E is path connected $\Longrightarrow E$ is connected.

- **Identity Theorem**

Let $f, g : I \to \mathbb{R}$ be analytic, where I is an open interval.
$E = \{x \in I : f(x) = g(x)\}$.
If E contains a cluster point of $E,$ then $f = g$ on I.

Stone-Weierstrass

- **Uniform metric space**

$$\text{Let } X \text{ be compact and } C(X) = \{f : X \to \mathbb{R} : f \text{ continuous}\}.$$
$$\rho(f,g) = \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|.$$
$$\text{Then } f_n \to f \text{ in } C(X) \iff f_n \to f \text{ uniformly on } X.$$

- **Algebra**

$$\text{Let } X \text{ be a set and } \mathcal{A} \text{ a set of functions from } X \text{ to } \mathbb{R}.$$
$$\mathcal{A} \text{ is an algebra if } f+g,\ fg,\ cf \in \mathcal{A} \quad \forall f,g \in \mathcal{A},\ \forall c \in \mathbb{R}.$$
$$\mathcal{A} \text{ separates points of } X \iff \forall x,y \in X,\ x \neq y,\ \exists f \in \mathcal{A} \text{ such that } f(x) \neq f(y).$$
$$\mathcal{A} \text{ vanishes at no point of } X \iff \forall x \in X,\ \exists f \in \mathcal{A} \text{ such that } f(x) \neq 0.$$
$$\mathcal{A} \text{ is an algebra} \implies \overline{\mathcal{A}} \text{ is an algebra.}$$

- **Stone-Weierstrass Theorem**

$$\text{Let } X \text{ be compact and } C(X) \text{ the uniform norm space.}$$
$$\mathcal{A} \subseteq C(X), \quad \mathcal{A} \text{ is an algebra, separates points of } X, \text{ and vanishes at no point of } X$$
$$\implies \overline{\mathcal{A}} = C(X).$$

- **Weierstrass Approximation Theorem**

$$f \in C([a,b]) \implies \exists \text{ a sequence of polynomials } p_n \to f \text{ uniformly on } [a,b].$$

Partial Derivatives

- **Partial Derivative**

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h\to 0} \frac{f(a+he_j) - f(a)}{h}, \qquad D_j f(a) = \frac{\partial f}{\partial x_j}(a).$$

- **C^r functions**

$$\text{Let } f : V \to \mathbb{R}^m, \quad V \subseteq \mathbb{R}^n, \quad r \in \mathbb{N}.$$
$$f \in C^r(V) \iff \text{all partial derivatives of } f \text{ of order } \le r \text{ exist and are continuous on } V.$$

- **Clairaut's Theorem**

$$\text{Let } f : V \to \mathbb{R}, \quad V \text{ open in } \mathbb{R}^n.$$
$$f \in C^2(V) \Longrightarrow \frac{\partial^2 f}{\partial x_j \partial x_k}(a) = \frac{\partial^2 f}{\partial x_k \partial x_j}(a).$$

- **Interchange of limits and integrals**

$$\text{Let } f_n : I \to \mathbb{R} \text{ be continuous and } f_n \to f \text{ uniformly on } I = [a,b].$$
$$\text{Then } f \text{ is continuous on } I, \qquad \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

- **Interchange of derivative and integral**

$$\text{Let } f : H \times \mathbb{R} \to \mathbb{R}, \quad H \times [a,b] \subseteq \mathbb{R}^2.$$
$$\text{If the relevant partial derivative is continuous, then } \frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

Linear Transformations

- **Notation**

$$A \in L(\mathbb{R}^n, \mathbb{R}^m), \qquad A = [a_{ij}]_{m\times n}.$$

- **Norm of matrix**

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

$$\|Ax\| \le \|A\| \, \|x\| , \qquad \|AB\| \le \|A\| \, \|B\| , \qquad \|A + B\| \le \|A\| + \|B\| .$$

$$\rho(A, B) = \|A - B\| \text{ is a metric on } L(\mathbb{R}^n, \mathbb{R}^m).$$

$$\|A\| \le \left(\sum_{i,j} a_{ij}^2 \right)^{1/2} .$$

- **Cauchy-Schwarz**

$$|u \cdot v| \le \|u\| \, \|v\| .$$

Differentiability

- **Total derivative**

Let $V \subseteq \mathbb{R}^n$ be open, $a \in V$.

f is differentiable at $a \iff \exists T \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - Th\|}{\|h\|} = 0.$$

$$Df(a) = T.$$

- **Continuous differentiability**

Let $V \subseteq \mathbb{R}^n$ be open and $f : V \to \mathbb{R}^m$.

If f is continuously differentiable at a, then f is differentiable at a.

If f is continuously differentiable on V, then $f \in C^1(V)$.

- **Continuously differentiable implies differentiable**

f is continuously differentiable at $a \implies f$ is differentiable at a.

- **Chain rule**

$$U \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m \text{ open}, \qquad f : U \to \mathbb{R}^m, \quad g : V \to \mathbb{R}^k.$$

$$f(U) \subseteq V, \quad f \text{ differentiable at } a, \quad g \text{ differentiable at } f(a)$$

$$\implies g \circ f \text{ differentiable at } a, \qquad D(g \circ f)(a) = Dg(f(a))Df(a).$$

Mean Value Theorem

- **Line segment**

Let $x, a \in \mathbb{R}^n$.

The line segment between x and a is $L(x, a) = \{a + t(x - a) : 0 \le t \le 1\}$.

- **Convexity**

$E \subseteq \mathbb{R}^n$ is convex $\iff L(x, y) \subseteq E, \quad \forall x, y \in E$.

- **Scalar function MVT**

$$\text{Let } f : V \to \mathbb{R} \text{ be differentiable on } V, \quad V \subseteq \mathbb{R}^n \text{ open.}$$
$$V \text{ convex}, \ x, a \in V \implies \exists c \in L(x, a) \text{ such that}$$
$$f(x) - f(a) = Df(c)(x - a) = \nabla f(c) \cdot (x - a).$$

- **Vector-valued MVT**

$$\text{Let } f : V \to \mathbb{R}^m \text{ be differentiable on } V, \quad V \subseteq \mathbb{R}^n \text{ open.}$$
$$\forall x, a \in V \text{ with } L(x, a) \subseteq V, \quad \exists c \in L(x, a) \text{ such that}$$
$$\|f(x) - f(a)\| \leq \|Df(c)\| \, \|x - a\| \, .$$

- **Continuity of derivative map**

$$\text{Let } V \subseteq \mathbb{R}^n \text{ be open and } f : V \to \mathbb{R}^m.$$
$$f \in C^1(V) \iff Df : V \to L(\mathbb{R}^n, \mathbb{R}^m) \text{ is continuous.}$$

- **Bound on compact and convex set**

$$\text{Let } f : V \to \mathbb{R}^m \text{ be } C^1 \text{ on } V, \quad V \subseteq \mathbb{R}^n \text{ open.}$$
$$\text{If } K \subseteq V \text{ is compact and convex, then } \exists M \geq 0 \text{ such that}$$
$$\|f(x) - f(a)\| \leq M \, \|x - a\| \, , \qquad \forall x, a \in K.$$

Inverse Function Theorem and Implicit Function Theorem

- **Invertible**

$$\text{Let } B \in L(\mathbb{R}^n), \quad A \text{ is invertible.}$$
$$A+B \text{ and } B \in L(\mathbb{R}^n) \text{ with } \|B-A\| \le \frac{1}{2\left\|A^{-1}\right\|} \Longrightarrow A+B \text{ is invertible and } \left\|(A+B)^{-1}\right\| \le 2\left\|A^{-1}\right\|.$$
$$A \mapsto A^{-1} \text{ is a continuous map.}$$

- **C-Lipschitz**

$$f : X \longrightarrow X \text{ is } C\text{-Lipschitz if } C < 1 \text{ and } \rho(f(x), f(y)) \le C\rho(x,y), \quad \forall x, y \in X.$$

- **Contraction Principle**

$$\text{Let } (X, \rho) \text{ be nonempty and complete metric space.}$$
$$\text{If } f : X \longrightarrow X \text{ is } C\text{-Lipschitz for } C < 1, \text{ then it is a contraction and } \exists! x \in X \text{ such that } f(x) = x.$$

- **Inverse Function Theorem**

$$\text{Let } V \subseteq \mathbb{R}^n \text{ be open, } a \in V, \quad f : V \longrightarrow \mathbb{R}^n \text{ be } C^1 \text{ on } V, \quad Df(a) \text{ invertible.}$$
$$\Longrightarrow \exists U \ni a,\ W \ni f(a) \text{ open such that } f : U \longrightarrow W \text{ is one-to-one and onto,}$$
$$f^{-1} : W \longrightarrow U \text{ is } C^1 \text{ and } D(f^{-1})(f(x)) = (Df(x))^{-1}, \quad \forall x \in U.$$

- **Partial Jacobian**

$$\text{Let } F = (f_1, \ldots, f_m) : U \longrightarrow \mathbb{R}^m, \quad U \subseteq \mathbb{R}^{n+m} \text{ open,} \quad (x_0, y_0) \in U.$$
$$D_y F = \left[\frac{\partial f_i}{\partial y_j}\right]_{m\times m}, \qquad D_x F = \left[\frac{\partial f_i}{\partial x_j}\right]_{m\times n}.$$

- **Implicit Function Theorem**

$$\text{Let } U \subseteq \mathbb{R}^{n+m} \text{ be open, } (a,b) \in U,\ c \in \mathbb{R}^m, \quad f : U \longrightarrow \mathbb{R}^m \text{ is } C^1 \text{ on } U.$$
$$f(a,b) = c, \qquad D_y f(a,b) \text{ is invertible.}$$
$$\Longrightarrow \exists V \ni a,\ W \ni b \text{ open and } g : V \longrightarrow W \text{ such that for each } x \in V$$
$$\exists! y \in W \text{ with } (x,y) \in U \text{ and } f(x,y) = c, \qquad y = g(x).$$
$$g \in C^1(V), \qquad f(x, g(x)) = c, \qquad Dg(x) = -[D_y f(x, g(x))]^{-1} D_x f(x, g(x)).$$

Riemann-Lebesgue Theorem

- **Upper and lower sums**

$$\text{Let } \mathcal{P} \text{ be subrectangles of } P.$$
$$U(f,\mathcal{P}) = \sum_{M\in\mathcal{P}} M_f(M)v(M), \qquad L(f,\mathcal{P}) = \sum_{M\in\mathcal{P}} m_f(M)v(M),$$
$$M_f(M) = \sup_{x\in M} f(x), \qquad m_f(M) = \inf_{x\in M} f(x).$$
$$\int_A f = \sup_{\mathcal{P}} L(f,\mathcal{P}), \qquad \int_A f = \inf_{\mathcal{P}} U(f,\mathcal{P}).$$

- **Measure zero**

$$E \text{ has measure zero in } \mathbb{R}^n \iff \forall\varepsilon > 0,\ \exists \text{ a countable cover } \{A_j\}_{j=1}^{\infty} \text{ of } E \text{ by closed/open rectangles}$$
$$\text{such that } \sum_{j=1}^{\infty} v(A_j) < \varepsilon.$$

- **Content zero**

$$E \text{ has content zero if } \forall\varepsilon > 0,\ \exists \text{ a finite cover } \{A_1,\dots,A_N\} \text{ of } E$$
$$\text{by closed/open rectangles such that } \sum_{j=1}^{N} v(A_j) < \varepsilon.$$

- **Riemann-Lebesgue Theorem**

Let $f : A \longrightarrow \mathbb{R}$ be bounded, where A is a closed rectangle in $\mathbb{R}^n$.

f is integrable on $A \iff \{x \in A : f \text{ is discontinuous at } x\}$ has measure zero.

Jordan Measurability and Fubini

- **Characteristic function**

Let A be a closed rectangle in $\mathbb{R}^n$, $\quad B \subseteq A$.

$$\chi_B : A \longrightarrow \mathbb{R} \text{ is the characteristic function,} \qquad \chi_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \in A \setminus B. \end{cases}$$

- **Jordan-measurable**

$$E \subseteq \mathbb{R}^n \text{ is Jordan-measurable if } E \text{ is bounded and } \partial E \text{ has measure } 0.$$

- **Integrability of**

χ_B

$$\text{If } E \subseteq A \text{ is Jordan-measurable for a closed rectangle } A \subseteq \mathbb{R}^n, \text{ then } \chi_E \text{ is integrable.}$$

$$\int_A \chi_B = \int_B 1.$$

$$E \text{ has content zero} \iff E \text{ is Jordan-measurable and } \int_A \chi_E = 0.$$

$$E \text{ has measure zero and } E \text{ is Jordan-measurable} \iff E \subseteq A \text{ for some closed rectangle } A \text{ and } \int_A \chi_E = 0.$$

- **Fubini's Theorem**

$$\text{Let } A \subseteq \mathbb{R}^n,\ B \subseteq \mathbb{R}^m \text{ be closed rectangles,} \quad f : A \times B \to \mathbb{R} \text{ be integrable on } A \times B.$$

$$\int_{A\times B} f = \int_A \left(\int_B f(x,y)\,dy\right) dx = \int_B \left(\int_A f(x,y)\,dx\right) dy.$$

$$\text{In particular, if for each } x \in A,\ \int_B f(x,y)\,dy \text{ exists, and for each } y \in B,\ \int_A f(x,y)\,dx \text{ exists,}$$

$$\text{then } \int_{A\times B} f = \int_A \int_B f(x,y)\,dy\,dx = \int_B \int_A f(x,y)\,dx\,dy.$$