MATH 320-2: Real Analysis II Notes

Series, Uniform Convergence, and Metric Spaces

Series Convergence

- **Convergence of series**

$$\text{Let } s_n = \sum_{k=1}^{n} a_k, \qquad \sum_{k=1}^{\infty} a_k \text{ converges if } \{s_n\} \text{ converges.}$$

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n.$$

- **Divergence test**

$$\text{If } a_k \not\to 0, \text{ then } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

- **Geometric series test**

$$\sum_{k=0}^{\infty} ar^k \text{ converges} \iff |r| < 1, \qquad \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

- **Cauchy Criterion**

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ s.t.}$$

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon \qquad \forall n \geq m \geq N.$$

- **Tail converges to 0**

$$\sum_{k=1}^{\infty} a_k \text{ converges} \implies \forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ s.t.}$$

$$\left| \sum_{k=n}^{\infty} a_k \right| < \varepsilon \ \forall n \geq N, \qquad \sum_{k=n}^{\infty} a_k \to 0 \text{ as } n \to \infty.$$

- **Convergence iff absolutely finite**

$$a_k \geq 0, \ \forall k \in \mathbb{N}.$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \{S_n\} \text{ converges} \iff \{S_n\} \text{ is bounded above.}$$

- **p-series test**

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \iff p > 1.$$

- **Comparison test**

$$0 \le a_k \le b_k \text{ for large } k.$$
$$\sum b_k \text{ conv} \implies \sum a_k \text{ conv}, \qquad \sum a_k \text{ div} \implies \sum b_k \text{ div}.$$

- **Limit Comparison Test**

$$a_k \ge 0,\ b_k > 0 \text{ for large } k, \qquad \lim_{k\to\infty} \frac{a_k}{b_k} = L.$$
$$\text{i) } L > 0: \quad \sum a_k \text{ conv} \iff \sum b_k \text{ conv}.$$
$$\text{ii) } L = 0: \quad \sum b_k \text{ conv} \implies \sum a_k \text{ conv}, \quad \sum a_k \text{ div} \implies \sum b_k \text{ div}.$$

- **Abel's formula**

$$\forall n \ge m \ge 1, \qquad \sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}).$$

- **Dirichlet test**

$$\text{If } \{A_n\} \text{ is bounded and } b_k \to 0, \text{ then } \sum_{k=1}^{\infty} a_k b_k \text{ converges.}$$

- **Alternating series test**

$$\text{If } b_k \ge 0 \text{ and } b_k \to 0, \text{ then } \sum_{k=1}^{\infty} (-1)^k b_k \text{ converges.}$$

Absolute Convergence

- **Absolute Convergence**

$$\sum a_k \text{ converges absolutely} \iff \sum |a_k| \text{ converges} \implies \sum a_k \text{ converges.}$$
$$\sum a_k \text{ converges conditionally} \iff \sum |a_k| \text{ diverges and } \sum a_k \text{ converges.}$$

- **Rearrangement**

$$\sum_{j=1}^{\infty} b_j \text{ is a rearrangement of } \sum_{k=1}^{\infty} a_k$$
$$\text{if } \exists \text{ a bijection } \phi : \mathbb{N} \to \mathbb{N} \text{ s.t. } a_k = b_{\phi(k)}.$$

- **Rearrangement Thm**

$$\text{If } \sum a_k \text{ converges absolutely, and } \sum b_j \text{ is any rearrangement of } \sum a_k,$$
$$\text{then } \sum b_j = \sum a_k \text{ so it converges.}$$

- **Limsup**

$$\text{Let } \{x_n\} \text{ be a sequence in } \mathbb{R},\ x \in \mathbb{R}.$$
$$\text{i) } \lim x_n = x \iff x_n < x \text{ for large } n.$$
$$\text{ii) } \limsup x_n \le x \implies x_n < x \text{ for infinitely many } n.$$
$$\text{iii) } x_n \to x \implies \limsup x_n = x.$$
$$\text{iv) } x_n \to -\infty \implies \limsup x_n = -\infty.$$
$$\text{v) } x_n \to \infty \implies \limsup x_n = \infty.$$

- **Root test**

$$\text{Let } r = \limsup_{k\to\infty} |a_k|^{1/k} \in \mathbb{R} \cup \{\infty\}.$$
$$r < 1 \implies \sum a_k \text{ conv abs,} \qquad r > 1 \implies \sum a_k \text{ div,} \qquad r = 1 \implies \text{inconclusive.}$$

- **Ratio test**

$$\text{If } a_k \neq 0 \text{ for large } k, \quad r = \limsup_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|.$$
$$r < 1 \implies \sum a_k \text{ conv abs,}$$

$$\left|\frac{a_{k+1}}{a_k}\right| \geq 1 \text{ for large } k \Longrightarrow \sum a_k \text{ div.}$$

$$\text{If } R = \lim\left|\frac{a_{k+1}}{a_k}\right|, \quad R < 1 \Longrightarrow \sum a_k \text{ conv abs}, \quad R > 1 \Longrightarrow \sum a_k \text{ div}, \quad R = 1 \Longrightarrow \text{inconclusive.}$$

Uniform Convergence

- **Pointwise Convergence**

$$\text{Let } E \neq \emptyset,\ f_n : E \to \mathbb{R}.$$
$$\text{Say } f_n \to f \text{ pointwise on } E$$
$$\text{if } \forall x \in E,\ \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ |f_n(x) - f(x)| < \varepsilon.$$

- **Uniform Convergence**

$$\text{Say } f_n \to f \text{ uniformly on } E$$
$$\text{if } \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ |f_n(x) - f(x)| < \varepsilon \quad \forall x \in E.$$

- **Uniform convergence lemma**

$$\text{If } \exists a_n \in \mathbb{R} \text{ s.t. } |f_n(x) - f(x)| \leq a_n \quad \forall x \in E,$$
$$\text{and } a_n \to 0,\ \text{then } f_n \to f \text{ uniformly on } E.$$

- **Uniform Convergence preserves boundedness**

$$\text{If } f_n \to f \text{ uniformly on } E,\ \text{each } f_n \text{ bounded on } E,$$
$$\text{then } f \text{ is bounded on } E.$$

- **Reverse Triangle Inequality**

$$\big||x| - |y|\big| \leq |x - y|, \qquad |x| + |y| \leq |x - y| \quad \text{(as recorded)}.$$

- **Uniform convergence preserves continuity**

$$\text{Let } \emptyset \neq E \subseteq \mathbb{R},\ a \in E.$$
$$\text{If } f_n \to f \text{ uniformly on } E,\ \text{and each } f_n \text{ is continuous at } a,$$
$$\text{then } f \text{ is continuous at } a.$$

- **Uniform Cauchy Criterion**

$$\{f_n\} \text{ converges uniformly on } E \iff \{f_n\} \text{ is uniformly Cauchy on } E$$
$$\iff \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } |f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \geq N,\ \forall x \in E.$$

- **Uniform convergence preserves integrals**

$$\text{Let } a < b \text{ in } \mathbb{R},\ f_n \to f \text{ uniformly on } [a,b].$$
$$\text{If each } f_n \text{ is integrable, then } f \text{ is integrable on } [a,b],$$
$$\int_a^b f_n \to \int_a^b f, \qquad \int_a^x f_n \to \int_a^x f \text{ uniformly for } x \in [a,b].$$

- **Limit swapping**

$$g_n \to g \text{ uniformly and each } g_n \text{ continuous at } c \in E \implies g \text{ continuous at } c.$$
$$\lim_{x\to c}\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} g_n(c) = \lim_{n\to\infty}\lim_{x\to c} g_n(x).$$

- **Uniform convergence and differentiation**

$$\text{Let } a < b,\ \text{each } f_n \text{ diff'ble on } [a,b].$$
$$\text{If } f_n' \text{ converges uniformly on } (a,b) \text{ and } f_n(x_0) \text{ converges for some } x_0 \in (a,b),$$
$$\text{then } f_n \text{ converges uniformly and } \left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} f_n' \text{ on } (a,b).$$

- **Weierstrass M-test**

$$f_k : E \to \mathbb{R}, \quad \forall k \in \mathbb{N}, \quad \exists M_k \in \mathbb{R} \text{ s.t. } |f_k(x)| \le M_k\ \forall x \in E.$$
$$\text{If } \sum_{k=1}^{\infty} M_k \text{ converges, then } \sum_{k=1}^{\infty} f_k \text{ converges absolutely and uniformly on } E.$$

- **Swapping series**

$$\text{If } \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} \text{ converges absolutely, then } \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij}.$$

Power Series

- **Power series definition**

A power series centered at $x_0 \in \mathbb{R}$ with radius of convergence R

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \qquad a_k \in \mathbb{R}.$$

$f(x)$ converges absolutely for $|x - x_0| < R$, $\quad f(x)$ diverges for $|x - x_0| > R$.

$f(x)$ converges uniformly on $[a, b] \subset (x_0 - R, x_0 + R)$.

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} \quad \text{or} \quad \frac{1}{|q|} = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| \text{ when the limit exists.}$$

- **Continuity**

f is continuous on $(x_0 - R, x_0 + R)$ and on any $[a, b] \subset (x_0 - R, x_0 + R)$.

- **Term by term differentiation**

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}, \qquad x \in (x_0 - R, x_0 + R).$$

$$f \in C^\infty(x_0 - R, x_0 + R), \qquad f^{(n)}(x_0) = n! a_n.$$

- **Abel's Thm**

If $f(x)$ converges at $x_0 + R$ or $x_0 - R$,

then f converges uniformly on $[x_0, x_0 + R]$ or $[x_0 - R, x_0]$,

and f is continuous at the endpoint.

- **Term by term integration**

If f converges on $[a, b]$, then f is integrable on $[a, b]$

and we have term by term integration.

Analytic Function

- **Multiplying Series Thm**

$$\text{Let } a_k, b_k \in \mathbb{R}, \qquad c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

$$\text{If } \sum a_k \text{ and } \sum b_k \text{ converge and one converges absolutely,}$$

$$\text{then } \sum c_k = \left(\sum a_k\right)\left(\sum b_k\right) \text{ converges.}$$

$$\text{If } \sum a_k, \sum b_k, \sum c_k \text{ converge on } (-R, R), \text{ then } \sum c_k x^k = \left(\sum a_k x^k\right)\left(\sum b_k x^k\right).$$

- **Analytic function**

Locally expressible as a power series.

$$f \text{ is analytic on } (a, b) \implies \forall x_0 \in (a, b), \ \exists \text{ open interval } (c, d) \subset (a, b)$$

$$f \in C^\infty(a, b) \text{ w/ } x_0 \in (c, d) \text{ s.t.}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \qquad \forall x \in (c, d).$$

- **Analytic M-test**

$$\text{Let } f \in C^\infty(a, b), \text{ if } \exists M > 0 \text{ s.t. } |f^{(n)}(x)| \le M^n, \quad \forall x \in (a, b), \ \forall n \in \mathbb{N},$$

$$\implies f \text{ is analytic on } (a, b).$$

- **A power series is Analytic**

A power series is analytic on its interval of convergence $(-R, R)$.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \text{for } x \text{ w/ } |x - x_0| < R - |x_0|.$$

- **Identity Thm**

$$\text{If } f \text{ and } g \text{ are analytic on } (a, b) \text{ and } f = g \text{ on } (c, d) \subset (a, b),$$

$$\implies f = g \text{ on } (a, b).$$

Metric Spaces

- **Metric Definition**

$$\text{Let } X \text{ be a set, } \rho : X \times X \to \mathbb{R} \text{ is a metric such that}$$
$$\rho(x,y) \geq 0, \quad \rho(x,y) = 0 \iff x = y, \quad \rho(x,y) = \rho(y,x),$$
$$\rho(x,y) \leq \rho(x,z) + \rho(z,y).$$

- **Open and closed balls**

$$B_r(x) = \{y \in X : \rho(x,y) < r\}, \qquad \overline{B}_r(x) = \{y \in X : \rho(x,y) \leq r\}.$$

- **Open and closed metric spaces**

$$V \text{ is open in } X \iff \forall x \in V,\ \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq V.$$
$$E \text{ is closed in } X \iff E^c \text{ is open in } X.$$
$$\text{Sequential characterization of closed sets: } \forall x_n \to x \text{ w/ each } x_n \in E \implies x \in E.$$

- **Completeness**

$$\text{A metric space } (X,\rho) \text{ is complete if every Cauchy sequence converges in } (X,\rho).$$

- **Sequential compact**

$$E \text{ is sequentially compact} \iff \text{every bounded sequence in } E \text{ has a subsequence converging in } E.$$

Limit of Functions

- **Cluster point**

$$\text{Let } (X,\rho) \text{ be a metric space, } E \subseteq X,\ a \in X.$$
$$a \text{ is a cluster point of } E \iff \forall \delta > 0,\ B_\delta(a) \cap E \text{ contains infinitely many points}$$
$$\iff \forall \delta > 0,\ B_\delta(a) \cap E \setminus \{a\} \neq \varnothing$$
$$\iff \exists \text{ sequence } \{x_n\} \text{ in } E \setminus \{a\} \text{ s.t. } x_n \to a.$$

- **Continuity**

$$E \subseteq X, \quad f : E \to Y.$$

$$f \text{ is continuous at } a \in E \iff \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } \tau(f(x), f(a)) < \varepsilon \quad \forall x \in E \text{ w/ } \rho(x,a) < \delta.$$

- **SCL**

$$\lim_{x \to a} f(x) = L \iff \forall \text{ sequences } x_n \text{ w/ each } x_n \in E \text{ s.t. } x_n \to a,\ f(x_n) \to L.$$

- **SCC**

$$f \text{ is continuous at } a \in E \iff \forall \text{ sequences } x_n \text{ w/ each } x_n \in E,\ x_n \to a, \text{ then } f(x_n) \to f(a).$$

- **Continuous functions**

The coordinate function $\mathbb{R}^n \to \mathbb{R}, \quad x = (x_1, \dots, x_n) \mapsto x_j$ is continuous.

Polynomials are continuous on $\mathbb{R}^n$, every rational of polynomial is continuous on $\mathbb{R}^n$.

Composite of continuous functions are continuous.

Interior, Closure, Boundary

- **Set union and intersection**

Let (X, ρ) be a metric space, A be an index set.

$$\text{Each } V_\alpha \text{ is open in } X \implies \bigcup_{\alpha \in A} V_\alpha \text{ is open.}$$

$$\text{Each } V_k \text{ is open in } X,\ k = 1, \dots, n \implies \bigcap_{k=1}^{n} V_k \text{ is open.}$$

$$\text{Each } E_\alpha \text{ is closed in } X \implies \bigcap_{\alpha \in A} E_\alpha \text{ is closed.}$$

$$\text{Each } E_k \text{ is closed in } X,\ k = 1, \dots, n \implies \bigcup_{k=1}^{n} E_k \text{ is closed.}$$

- **Interior, Closure, Boundary**

$$\operatorname{int} E = \{x \in E : \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq E\}.$$

$$\text{Closure: } x \in \overline{E} \iff \forall r > 0,\ B_r(x) \cap E \neq \varnothing \iff x \text{ is a limit point or } x \in E.$$

$$\text{Boundary: } x \in \partial E \iff \forall r > 0,\ B_r(x) \cap E \neq \varnothing \text{ and } B_r(x) \cap E^c \neq \varnothing.$$

$$E^\circ \text{ is open}, \quad \overline{E} \text{ is closed}, \quad \partial E \text{ is closed}, \quad \overline{E} = E^\circ \cup \partial E.$$

- **Closure**

$$A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}.$$

$$\overline{E} \text{ is closed,} \qquad F \text{ is closed and } E \subseteq F \Longrightarrow \overline{E} \subseteq F.$$

$$\overline{E} = \bigcap \{F : F \text{ is closed and } E \subseteq F\}.$$

$$\overline{A \cup B} = \overline{A} \cup \overline{B}, \qquad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

- **Duality and Boundary**

$$(\overline{E})^c = (E^c)^\circ, \qquad (E^\circ)^c = \overline{E^c}.$$

$$\partial E = \overline{E} \cap \overline{E^c} = \overline{E} \setminus E^\circ.$$