MATH 320-1: Midterm 2 Notes

Real Analysis I

- **Lim Sup and Lim Inf:**

Let x_n be bounded, then

- $$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k = \inf_{n\in\mathbb{N}} \sup_{k\geq n} x_k \qquad e_{x_i}, i \in J$$

- $$\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \inf_{k\geq n} x_k = \sup_{n\in\mathbb{N}} \inf_{k\geq n} x_k$$

- **Limits of Functions:** ⋆

$$\lim_{x\to a} f(x) = L \quad \text{or} \quad f(\omega) \longrightarrow L \text{ as } x \longrightarrow a$$

$$\text{if } \forall\varepsilon > 0,\ \exists\delta > 0 \text{ s.t. } |f(x) - L| < \varepsilon \quad \forall x \in I \text{ w/ } 0 < |x - a| < \delta$$

- **Limits of sequence:** ⋆ **From Midterm 1**

$$\lim_{n\to\infty} x_n = a \quad \text{or} \quad x_n \longrightarrow a$$

$$\text{if } \forall\varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ |x_n - a| < \varepsilon$$

- **Sequential Characterization of Limits**

$$f(x) \longrightarrow L \text{ as } x \longrightarrow a \quad \Longleftrightarrow \quad f(x_n) \longrightarrow L \quad \forall \text{ sequences } x_n \longrightarrow a \text{ w/ each } x_n \in I \setminus \{a\}$$

- **Divergence Criteria**

If $\exists$ sequence $x_n \longrightarrow a$ w/ $x_n \in I \setminus \{a\}$

s.t. $\{f(x_n)\}$ diverges,

or $\exists$ sequences $x_n \longrightarrow a$ and $y_n \longrightarrow a$ w/ $x_n, y_n \in I \setminus \{a\}$

s.t. $\{f(x_n)\}$ and $\{f(y_n)\}$ converges to different limits,

then $\lim_{x\to a} f(x)$ DNE

Limits of functions

$$\text{Note: } \left\{\sin\left(\frac{\pi}{2} + n\pi\right)\right\} = \{(-1)^n\} \qquad \forall n \in \mathbb{N}$$

- **Algebraic Laws of limits of functions:**

$$\text{Suppose } \lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

$$\text{Then } \lim_{x\to a}(f+g)(x) = L + M, \qquad \lim_{x\to a}(\alpha f)(x) = \alpha L \quad \forall \alpha \in \mathbb{R}$$

$$\lim_{x\to a}(fg)(x) = LM, \qquad \lim_{x\to a} \frac{f}{g}(x) = \frac{L}{M} \quad \text{if } M \neq 0$$

- **Squeeze Thm:**

$$\text{If } f(x) \leq g(x) \leq h(x), \quad \forall x \in I \setminus \{a\},$$

$$\text{and } \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L, \quad \text{then } \lim_{x\to a} g(x) = L$$

- **Comparison Thm:**

$$\text{If } f(x) \leq g(x), \quad \forall x \in I \setminus \{a\},$$

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M,$$

$$\text{then } L \leq M$$

$$\text{Let } f : (a, b) \to \mathbb{R}, \text{ where } a < b$$

- **Right hand limit:**

$$\lim_{x\to a^+} f(x) = L \quad \text{if } \forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t.}$$

$$|f(x) - L| < \varepsilon \quad \forall x \in (a, b) \text{ w/ } a < x < a + \delta$$

- **Left hand limit:**

$$\lim_{x\to b^-} f(x) = L \quad \text{if } \forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t.}$$

$$|f(x) - L| < \varepsilon \quad \forall x \in (a, b) \text{ w/ } b - \delta < x < b$$

Continuity

Let $f : E \longrightarrow \mathbb{R}$, where $\varnothing \neq E \subseteq \mathbb{R}$

- **Continuous function:** ⋆

$$f \text{ is continuous at } a \in E \text{ if } \forall \varepsilon > 0,\ \exists \delta > 0$$
$$\text{s.t. } |f(x) - f(a)| < \varepsilon \quad \forall x \in E \text{ w/ } |x - a| < \delta$$

Say f is continuous on E if f is continuous at every $a \in E$

- **Sequential Characterization of Continuity** ⋆

$$\text{Let } f : E \longrightarrow \mathbb{R}, \text{ where } \varnothing \neq E \subseteq \mathbb{R},\ a \in E$$
$$\text{Then } f \text{ is continuous at } a$$
$$\iff \quad f(x_n) \longrightarrow f(a) \quad \forall \text{ sequences } x_n \longrightarrow a \text{ w/ each } x_n \in E$$

- **Algebraic Laws of Continuous functions**

$$\text{Let } f, g : E \longrightarrow \mathbb{R}, \text{ where } \varnothing \neq E \subseteq \mathbb{R},$$
$$a \in E, \text{ if } f \text{ and } g \text{ are continuous at } a, \text{ then so are}$$
$$f + g, \qquad \alpha f,\ \forall \alpha \in \mathbb{R}, \qquad fg, \qquad \frac{f}{g} \text{ if } g(a) \neq 0, \qquad \frac{1}{g}$$

Every polynomial is continuous, and $\dfrac{p(x)}{q(x)}$ (p, Q are polynomials)

- **Composite functions**

$$\text{Let } f : A \longrightarrow \mathbb{R}, \quad g : B \longrightarrow \mathbb{R}, \quad \text{where } A, B \subseteq \mathbb{R},\ f(A) \subseteq B$$
$$(g \circ f)(x) = g(f(x)), \quad x \in A$$
$$\text{if } f \text{ is continuous at } a \in A \text{ and } g \text{ is continuous at } f(a),$$
$$\text{then } g \circ f \text{ is continuous at } a$$

- **Let $f, g : E \to \mathbb{R}$, if f and g are continuous on E,**

$$\text{so are:}$$
$$|f|, \quad |g|, \quad |f+g|, \quad |f-g|, \quad \max\{f(x), g(x)\}$$

- **Extreme Value Thm ⋆ ⋆**

$$\text{Let } f : [a,b] \to \mathbb{R} \text{ be continuous, where } a \le b \text{ in } \mathbb{R}, \text{ then}$$
$$\text{i) } f \text{ is bounded i.e. } \exists M \in \mathbb{R} \text{ s.t. } |f(x)| \le M \quad \forall x \in [a,b]$$
$$\text{ii) } f \text{ attains a minimum and a maximum}$$

- **Approximation property for a sequence converging to supremum**

$$\text{Let } S \text{ be bounded and } \varnothing \ne S \subseteq \mathbb{R},$$
$$\text{Let } L = \sup(S), \text{ By the Approximation property,}$$
$$\forall \varepsilon > 0, \ \exists a \in S \text{ s.t. } L - \varepsilon < a \le L$$
$$\text{let } \varepsilon = \frac{1}{n}, \text{ we have } \{x_n\} \subseteq S \text{ s.t.}$$
$$L - \frac{1}{n} < x_n \le L, \quad \text{by squeeze, } \lim_{n\to\infty} x_n = L$$

- **Intermediate Value Thm ⋆ ⋆**

$$\text{Let } f : [a,b] \to \mathbb{R} \text{ be continuous, where } a < b \text{ in } \mathbb{R}$$
$$\text{If } f(a) < y_0 < f(b), \ \exists x_0 \in [a,b] \text{ s.t. } f(x_0) = y_0$$

- **Triy Ineq:**

$$|a+b| \le |a| + |b|$$
$$\big||a| - |b|\big| \le |a-b|$$

Uniform Continuity

- **Uniformly Continuous functions:**

$$\text{Let } f : E \to \mathbb{R},\ \text{ where } \varnothing \neq E \subseteq \mathbb{R}$$
$$f \text{ is uniformly continuous on } E \text{ if } \forall \varepsilon > 0,\ \exists \delta > 0$$
$$\text{s.t. } |f(x) - f(y)| < \varepsilon \quad \forall x, y \in E \text{ w/ } |x - y| < \delta$$
$$\star \quad \delta \text{ only depend on } \varepsilon$$

- **Uniform continuity and Cauchy**

$$\text{Let } f : E \to \mathbb{R} \text{ be uniformly continuous, where } \varnothing \neq E \subseteq \mathbb{R},$$
$$\text{if } \{x_n\} \text{ is Cauchy w/ } x_n \in E,\ \text{ then } \{f(x_n)\} \text{ is Cauchy}$$

- **Continuous to uniformly continuous Thm**

$$\text{Any continuous function } f : [a, b] \to \mathbb{R} \text{ is}$$
$$\text{uniformly continuous, where } a \leq b$$

Differentiability

- **Differentiable function**

$$\text{Let } a \in \mathbb{R},\ I \text{ an open interval containing } a,\ f : I \to \mathbb{R},$$
$$\text{Say } f \text{ is differentiable at } a \text{ if}$$
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
$$\text{exists, and } f \text{ is differentiable on } I \text{ if } f \text{ is differentiable}$$
$$\text{on every } a \in I. \quad f''(a) = (f')'(a),\ \text{ in general, } f^{(n)}(a) = \left(f^{(n-1)}\right)'(a),\ n \in \mathbb{N}$$

- **Continuously differentiable**

$$f \text{ is continuously differentiable on } I \text{ and write } f \in C^1(I)$$
$$\text{if } f' \text{ exists and is continuous on } I$$
$$C^n(I) = \{f : I \to \mathbb{R},\ f \text{ is } n \text{ times continuously differentiable}\} \quad n \in \mathbb{N}$$
$$C^\infty(I) = \{f : I \to \mathbb{R} :\ f \text{ is smooth or indefinitely differentiable, } f^{(n)} \text{ exists on } I\ \forall n \in \mathbb{N}\}$$

- **Differentiability on non-open intervals**

$$f : [a, b] \longrightarrow \mathbb{R},$$
$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$
$$f'(b) = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

- **Differentiability Implies Continuity Thm**

$$\text{If } f \text{ is differentiable at } a, \text{ then } f \text{ is continuous at } a$$

- **Chain Rule**

$$\text{Let } f : I \longrightarrow \mathbb{R}, \quad g : J \longrightarrow \mathbb{R}, \quad \text{where } I, J \text{ are open intervals}$$
$$\text{and } f(I) \subseteq J$$
$$\text{Suppose } f \text{ is diff'ble at } a \in I \text{ and } g \text{ is diff'ble at } f(a),$$
$$\text{Then } g \circ f \text{ is diff'ble at } a, \text{ and } (g \circ f)'(a) = g'(f(a))f'(a)$$

- **Algebraic Rules of differentiable functions**

$$\text{Let } f, g : I \longrightarrow \mathbb{R} \text{ be differentiable at } a, \text{ where } I \text{ is an interval containing } a,$$
$$\text{Then } f + g, \ \alpha f \ (\forall \alpha \in \mathbb{R}), \ fg, \ \frac{f}{g}(g(a) \neq 0) \text{ are all differentiable at } a$$
$$(f + g)'(a) = f'(a) + g'(a)$$
$$(\alpha f)'(a) = \alpha f'(a)$$
$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}, \qquad \left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{g^2(a)}$$

- **Power Rule**

$$(x^n)' = nx^{n-1}$$

Mean Value Theorem

- **Rolle's Thm:**

$$\text{Let } f : [a,b] \to \mathbb{R} \text{ be continuous on } [a,b] \text{ and differentiable on } (a,b)$$
$$\text{If } f(a) = f(b), \text{ then } \exists c \in (a,b)$$
$$\text{s.t. } f'(c) = 0$$

- **Mean Value Thm:**

$$\text{Let } f, g : [a,b] \to \mathbb{R} \text{ be continuous on } [a,b]$$
$$\text{and differentiable on } (a,b),$$
$$\text{then } \exists c \in (a,b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$
$$\text{more generally, } \exists c \in (a,b) \text{ s.t.}$$
$$f'(c)\big(g(b) - g(a)\big) = g'(c)\big(f(b) - f(a)\big)$$

- **strictly or non strictly increasing or decreasing**

$$\text{Let } f : [a,b] \to \mathbb{R} \text{ be cts on } [a,b]$$
$$\text{and diff'ble on } (a,b)$$
$$f \text{ is:}$$

• strictly increasing	if $f'(x) > 0$	$\forall x \in (a,b)$	$f(x) < f(y)$	if $x < y$
• increasing	if $f'(x) \geq 0$		$f(x) \leq f(y)$	
• strictly decreasing	if $f'(x) < 0$		$f(x) > f(y)$	
• decreasing	if $f'(x) \leq 0$		$f(x) \geq f(y)$	
• constant	if $f'(x) = 0$		$f(x) = f(y)$	

- **Intermediate Value Thm for derivatives**

$$\text{Let } f : [a,b] \to \mathbb{R} \text{ be diff'ble w/ } f'(a) \neq f'(b),$$
$$\text{If } f'(a) < y_0 < f'(b) \text{ or } f'(b) < y_0 < f'(a),$$
$$\text{then } \exists c \in [a,b] \text{ s.t. } f'(c) = y_0$$

- **Heine Cantor Thm**

$$\text{If } f : [a,b] \to \mathbb{R} \text{ is differentiable w/ } f'(x)$$
$$\text{being bounded on } (a,b), \text{ then } f \text{ is uniformly continuous}$$

- **Second derivative test**

$$\text{Let } a \in \mathbb{R}, \ I \text{ an open interval containing } a, \ f : I \to \mathbb{R} \text{ differentiable.}$$
$$\text{If } f'(a) = 0 \text{ and } f''(a) > 0, \text{ then } f \text{ has a local minimum}$$
$$\text{at } a, \text{ i.e. } \exists \delta > 0 \text{ w/ } (a-\delta, a+\delta) \subseteq I$$
$$\text{s.t. } f(x) \geq f(a) \quad \forall x \in (a-\delta, a+\delta)$$

- **Cauchy Sequences:**

$$\{x_n\} \text{ is Cauchy if } \forall \varepsilon > 0,$$
$$\exists N \in \mathbb{N} \text{ s.t. } |x_m - x_n| < \varepsilon, \quad \forall n, m \geq N$$