MATH 320-1: Final Notes

Real Analysis I

Taylor Polynomial

- **Taylor Polynomial**

Let f be n times differentiable at a.

Define nth polynomial of f centered at a to be $\quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$

and the nth Taylor remainder to be $\quad R_n(x) = f(x) - P_n(x).$

★ Condition: function can be approximated by polynomial.

- **Taylor's Thm**

Let $f : [a, b] \to \mathbb{R}$ be nth time differentiable, where $-\infty < a < b < \infty$, $n \in \mathbb{N} \cup \{0\}$.

Then $\forall x, x_0 \in (a, b)$, $\exists c$ between x and x_0 s.t.

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \;+\; \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

nth degree polynomial of f centered at x_0 $\qquad$ nth Taylor polynomial remainder.

Partitions

- **Partition definition**

i) A partition of $[a, b]$ is a finite set of points

$P = \{x_0, x_1, \dots, x_n\}$ s.t. $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

ii) Let $f : [a, b] \to \mathbb{R}$ be bounded. $\quad P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$.

Define the upper sum and lower sum as

$$U(f, P) = \sum_{k=1}^{n} \sup_{x \in [x_{k-1}, x_k]} f(x)(x_k - x_{k-1}),$$

$$L(f, P) = \sum_{k=1}^{n} \inf_{x \in [x_{k-1}, x_k]} f(x)(x_k - x_{k-1}).$$

Note: $L(f, P) \le U(f, P)$.

Let P, Q be partitions of $[a, b]$. If Q is a refinement of P,

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P).$$

Integration

- **Notation**

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, $a < b$ in $\mathbb{R}$.

Let P be the collection of all partitions of $[a, b]$.

- **Lower and upper integrals**

$$L(f, S) \leq u(f, Q) \qquad \forall S, Q \in P.$$

$$\sup_{S \in P} L(f, S) \leq \inf_{Q \in P} U(f, Q).$$

i) The upper integral of f on $[a, b]$ is $\overline{\int_a^b} f(x)\,dx = \inf_{P \in P} U(f, P)$.

ii) The lower integral of f on $[a, b]$ is $\underline{\int_a^b} f(x)\,dx = \sup_{P \in P} L(f, P)$.

Note: $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

iii) Say the function f is integrable on $[a, b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$.

- **Comparison Thm for integrals**

Let f, g be integrable on $[a, b]$, where $a < b$ in $\mathbb{R}$.

If $f(x) \leq g(x),\ \forall x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

In particular, if $m \leq f(x) \leq M,\ \forall x \in [a, b]$,

then $m(b - a) \leq \int_a^b f \leq M(b - a)$.

- **Approximation characterization of Integrability**

Let $f : [a, b] \to \mathbb{R}$ be bounded, with $a < b$ in $\mathbb{R}$,

and P be the set of all possible partitions of $[a, b]$.

f is integrable on $[a, b] \iff \forall \varepsilon > 0,\ \exists P_\varepsilon \in P$ s.t. $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

- **Integrability of continuous functions**

$$\text{Let } f : [a,b] \to \mathbb{R} \text{ be continuous, where } a < b \text{ in } \mathbb{R}, \text{ then } f \text{ is integrable on } [a,b].$$

- **Domain splitting**

$$\text{Let } f : [a,b] \to \mathbb{R} \text{ be bounded, where } a < b, \ c \in (a,b).$$
$$f \text{ is integrable on } [a,b] \iff f \text{ is integrable on } [a,c] \text{ and } [c,b].$$

- **Domain restriction**

$$\text{If } f \text{ is integrable on } [a,b], \text{ then it is integrable on } [c,d] \quad \forall c < d \text{ in } [a,b].$$

- **Endpoint don't matter**

$$\text{Let } f : [a,b] \to \mathbb{R} \text{ be integrable where } a < b \text{ in } \mathbb{R}.$$
$$\text{If } g : [a,b] \to \mathbb{R} \text{ satisfies } g(x) = f(x) \ \forall x \in (a,b),$$
$$\text{then } g \text{ is integrable on } [a,b] \text{ and } \int_a^b f = \int_a^b g.$$

- **Finitely many points don't matter**

$$\text{i) Let } f : [a,b] \to \mathbb{R} \text{ be integrable where } a < b.$$
$$\text{If } g : [a,b] \to \mathbb{R} \text{ satisfies } g(x) = f(x) \text{ for all but finitely many } x \in [a,b],$$
$$\text{then } g \text{ is integrable on } [a,b], \text{ and } \int_a^b f = \int_a^b g.$$
$$\text{ii) Let } f : [a,b] \to \mathbb{R} \text{ be bounded on } [a,b] \text{ and continuous at all but finitely many points,}$$
$$\text{then } f \text{ is integrable on } [a,b].$$

FTC

- **Fundamental Theorem of Calculus**

$$f : [a, b] \to \mathbb{R}.$$

$$\text{i) If } f \text{ is continuous on } [a, b] \text{ and } F(x) = \int_a^x f(t)\, dt,$$

$$\implies F \in C^1([a, b]), \qquad \frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = F'(x) = f(x) \text{ for each } x \in [a, b].$$

$$\text{ii) If } f \text{ is differentiable on } [a, b] \text{ and } f' \text{ is integrable on } [a, b],$$

$$\implies \int_a^x f'(t)\, dt = f(x) - f(a) \text{ for each } x \in [a, b].$$