Lead, Lag, and Stability Design via Bode Plots

Key Relationships

$$e_{ss} = \frac{1}{K_v}, \qquad K_v = \lim_{s \to 0} s \cdot D(s)P(s), \qquad \zeta \approx \frac{PM}{100}$$

$$\text{Lead magnitude boost at } \omega_c : \quad -20 \log_{10}(\sqrt{\alpha}) \text{ dB}$$

$$\text{Lag phase at crossover} \approx \arctan(n) - \arctan(n\beta) \approx -5 \text{ for } n = 10$$

Reading the Bode Plot

Identifying Poles at $s = 0$ (System Type)

- A pole at $s = 0$ (integrator) shows as a -20 dB/dec slope at low frequencies on the magnitude plot, and phase starting at -90 at low frequencies on the phase plot
- Each additional pole at $s = 0$ adds another -20 dB/dec to the low-frequency slope and another -90 to the low-frequency phase
- A Type 1 system (one integrator) has $K_v = \lim_{s \to 0} s \cdot G(s)$, read as the frequency where the low-frequency -20 dB/dec asymptote crosses 0 dB

Identifying RHP Zeros

- A RHP zero looks identical to a LHP zero in magnitude (+20 dB/dec slope contribution) but causes phase to **decrease** rather than increase
- If the phase plot drops where you expect a zero to cause a rise, the zero is in the RHP
- RHP zeros impose fundamental limits on achievable bandwidth and make control more difficult

Key Frequencies to Read from Bode Plot

- **Gain crossover frequency** ω_c: frequency where $|G(j\omega)| = 0$ dB
- **Phase crossover frequency** ω_{pc}: frequency where $\angle G(j\omega) = -180$
- **Phase margin**: $PM = 180 + \angle G(j\omega_c)$
- **Gain margin**: $GM = -20 \log_{10} |G(j\omega_{pc})|$ dB, i.e. how many dB the magnitude is below 0 dB at ω_{pc}

Stability Range for Constant Gain $D(s) = K$

For a unity feedback system with loop transfer function $K \cdot P(s)$, increasing K shifts the magnitude curve upward without affecting phase. The closed-loop system becomes unstable when the magnitude reaches 0 dB at the phase crossover frequency ω_{pc}.

Procedure

1. From the Bode plot of the **unscaled** $P(s)$ (with $K = 1$), read the phase crossover frequency ω_{pc} where:

$$\angle P(j\omega_{pc}) = -180$$

2. Read the magnitude at that frequency: $|P(j\omega_{pc})|$ in linear scale, or M_{pc} in dB
3. The closed-loop is stable for positive K satisfying:

$$K < \frac{1}{|P(j\omega_{pc})|} = 10^{-M_{pc}/20}$$

4. If the phase never reaches -180, the system is stable for all positive K (infinite gain margin)
5. If the phase starts below -180 at low frequencies and crosses -180 multiple times, check each crossing carefully using the Nyquist criterion

Gain and Phase Margin Interpretation

- **Gain margin** GM: how much you can multiply K before instability. A gain margin of X dB means K can increase by a factor of $10^{X/20}$ before the closed loop goes unstable
- **Phase margin** PM: how much additional phase lag the loop can tolerate before instability. Related to damping by $\zeta \approx PM/100$
- A good design typically requires $GM \geq 6$ dB and $PM \geq 30$

Controller Design for Steady-State Error (Part f type)

Step 0: Determine System Type from Bode Plot

- Count the number of integrators in $P(s)$ from the low-frequency Bode magnitude slope
- **Type 0** (flat low-frequency magnitude, phase starts at 0): step error is finite, ramp error is infinite. Need integrator in $D(s)$
- **Type 1** (-20 dB/dec low-frequency slope, phase starts at -90): ramp error is finite $= 1/K_v$. Gain adjustment or lag sufficient
- **Type 2** (-40 dB/dec, phase starts at -180): ramp error is zero

Step 1: Compute Required K_v

From the steady-state error spec $|e_{ss}| < e^*$:

$$K_v \geq \frac{1}{e^*}$$

For a Type 1 plant $P(s)$, $K_v = D(0) \cdot K_{v,plant}$ where $K_{v,plant} = \lim_{s \to 0} s \cdot P(s)$ is read from the Bode plot as the frequency where the low-frequency -20 dB/dec asymptote crosses 0 dB.

Step 2: Check if Pure Gain Suffices

Compute the required $D(0) = K$:

$$K \geq \frac{1}{e^* \cdot K_{v,plant}}$$

Check if this K keeps the closed loop stable using the gain margin from Step 0 of the stability section above:

- If $K < 1/|P(j\omega_{pc})|$: pure gain controller works. Set $D(s) = K$ and verify stability
- If $K \geq 1/|P(j\omega_{pc})|$: pure gain will destabilize the system. Need a lag compensator to boost K_v without pushing gain crossover into unstable region

Step 3a: Pure Gain Controller (if sufficient)

$$D(s) = K = \frac{1}{e^* \cdot K_{v,plant}}$$

Verify: $K < 1/|P(j\omega_{pc})|$ and $PM > 0$.

Step 3b: Lag Controller (if pure gain insufficient)

Use the lag design procedure. The plant already being Type 1 means you do not need to add an integrator — just boost the DC gain via β:

1. Target $PM_{target} = PM_{spec} + 5$ to account for phase lag added by compensator. If no PM spec given, target $PM \geq 40$ for reasonable damping

2. Find ω_c on Bode plot of unscaled $P(s)$ where:

$$\angle P(j\omega_c) = -(180 - PM_{target})$$

3. Read magnitude M dB at ω_c. Compute K:

$$K = 10^{-M/20}$$

4. Compute required lag ratio:

$$\beta = \frac{1}{K \cdot e^* \cdot K_{v,plant}}$$

5. Choose placement factor n between 5 and 10. Compute break frequencies:

$$z = \frac{\omega_c}{n}, \qquad p = \frac{z}{\beta}$$

6. Lag controller:

$$D(s) = K \cdot \frac{s/z + 1}{s/p + 1}$$

7. Verify: $PM \geq PM_{spec}$ and $K_v = K \cdot \beta \cdot K_{v,plant} \geq 1/e^*$

Method 2: Phase-First Lead Design

Use when a damping ratio spec is also given alongside the steady-state error spec.

Step 1: Find ω_c from Phase Condition

Plot Bode of unscaled $G(s)$ with $K = 1$. Choose $\alpha = p/z$ to provide required phase boost:

$$\phi_{max} = \arcsin\left(\frac{\alpha - 1}{\alpha + 1}\right) \geq PM_{spec} - \angle G(j\omega_c) - 180$$

Read ω_c where phase equals $-(180 - PM_{spec} - \phi_{max})$.

Step 2: Solve for K Analytically

$$z = \frac{\omega_c}{\sqrt{\alpha}}, \qquad p = \omega_c\sqrt{\alpha}$$

$$K = \frac{|j\omega_c/p + 1|}{|j\omega_c/z + 1| \cdot |G(j\omega_c)|}$$

Step 3: Check if Lead Alone Sufficient

- If $K \geq 1/e^*$: lead alone sufficient
$$D(s) = K\,\frac{s/z+1}{s/p+1}$$
- If $K < 1/e^*$: add lag on top with $\beta = 1/(K \cdot e^*)$, $z_{lag} = \omega_c/n$, $p_{lag} = z_{lag}/\beta$
$$D(s) = K\,\frac{s/z+1}{s/p+1} \cdot \frac{s/z_{lag}+1}{s/p_{lag}+1}$$