Dynamical Systems

Eric Yang

Definitions

Module 1

Definition 1.0. A *dynamical system* is the pair (X, f), with X being the set of all possible states, i.e. the phase space, and $f : X \to X$ a function.

Definition 1.1. An *orbit* of $x_0 \in X$ is the set:

$$\{x_0,\ f(x_0),\ f^2(x_0),\ f^3(x_0),\ \dots,\ f^n(x_0),\ \dots\}$$

Definition 1.2. A *fixed point* $p \in X$ satisfies $f(p) = p$.

Definition 1.3. An *eventually fixed point* $p \in X$ satisfies $f(f^n(p)) = f^n(p)$ for some $n \in \mathbb{N}$.

Definition 1.4. $p \in X$ is an *attracting fixed point* if $\exists\, \varepsilon > 0$ s.t. $\forall\, x \in X$ with $|x - p| < \varepsilon$,

$$\lim_{n \to \infty} f^n(x) = p.$$

Definition 1.5. $p \in X$ is a *repelling fixed point* if $\exists\, \varepsilon > 0$ s.t. $\forall\, x \in X$ with $|x - p| < \varepsilon$ and $x \neq p$,

$$\exists\, k \in \mathbb{N} \text{ s.t. } |f^k(x) - p| \geq \varepsilon.$$

Definition 1.6. The *basin of attraction* of an attracting fixed point $p \in X$ is the set of all $x_0 \in X$ s.t. the sequence $\{f^n(x_0)\} \to p$.

Module 3

Definition 3.1. Let $f : X \to X$ and $n \in \mathbb{N}$. A point $x \in X$ is a *periodic point of period n* for f if $f^n(x) = x$. If n is the smallest positive integer for which $f^n(x) = x$, we say x has *minimal period n*. The orbit of a periodic point with minimal period n is called an *n-cycle.* A point is called *eventually periodic* if one of its iterates is periodic.

Definition 3.3. Let x be a periodic point of minimal period n for f. We say that x is an *attracting periodic point* for f if it is an attracting fixed point for f^n. We say that x is a *repelling periodic point* for f if it is a repelling fixed point for f^n.

Definition 3.5. Let $\{p, f(p), \ldots, f^{n-1}(p)\}$ be an n-cycle for the dynamical system (X, f). We say it is a *repelling n-cycle* if p is a repelling periodic point for f, and it is an *attracting n-cycle* if p is an attracting periodic point for f.

Module 8

Definition 8.1. Let (X, f) and (Y, g) be dynamical systems. A function $\phi : X \to Y$ *commutes with f and g* if $\phi(f(x)) = g(\phi(x))$ for every $x \in X$. This is illustrated in the diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\
Y \arrow[r, "g"] & Y.
\end{tikzcd}$$

Definition 8.4. A function $\phi : X \to Y$ is a *homeomorphism* if it is one-to-one, onto, continuous, and its inverse function ϕ^{-1} is continuous.

Definition 8.5. The dynamical systems (X, f) and (Y, g) are *conjugate* if there exists a homeomorphism $\phi : X \to Y$ that commutes with f and g. We call the function ϕ a *conjugacy.*

Module 13

Definition 13.1. Let $\mathcal{A}$ be a finite set. The set $\mathcal{A}$ is called an *alphabet.*

Definition 13.2. The *full shift space over $\mathcal{A}$* is

$$\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}.$$

Definition 13.3. The *shift map* on $\mathcal{A}^{\mathbb{Z}}$ is the function

$$\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$$

defined by

$$(\sigma(x))_i = x_{i+1}$$

for every $i \in \mathbb{Z}$.

Definition 13.4. The symbolic dynamical system

$$(\mathcal{A}^{\mathbb{Z}}, \sigma)$$

is called the *full shift over* $\mathcal{A}$. If $\mathcal{A} = \{0, 1, \ldots, n-1\}$, then $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the *full shift on* n *symbols*.

Definition 13.5. Let $a, b \in \mathbb{N}$ with $a \leq b$. The (a, b)*-run length limited shift* is the set of all sequences in $\{0, 1\}^{\mathbb{Z}}$ such that between any two consecutive occurrences of 1, there are at least a and at most b occurrences of 0.

Definition 13.6. A *block* or *word* over an alphabet $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$.

Definition 13.7. The *length* of a block u is the number of symbols contained in u. A block of length k is called a k*-block*.

Definition 13.8. Let $x \in \mathcal{A}^{\mathbb{Z}}$ and let u be a block over $\mathcal{A}$. We say that u *occurs in* x if there exist indices $i \leq j$ such that

$$x_i x_{i+1} \cdots x_j = u.$$

Definition 13.9. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and let u be a block over $\mathcal{A}$. We say that u *occurs in* X if there exists some $x \in X$ such that u occurs in x.

Definition 13.10. Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$. The *shift space given by* $\mathcal{F}$ is the subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ defined by

$$X = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{no block from } \mathcal{F} \text{ occurs in } x\}.$$

Definition 13.11. A *symbolic dynamical system* is a pair (X, σ) where X is a shift space given by a collection $\mathcal{F}$ of forbidden blocks and σ is the shift map.

Definition 13.12. The set of *one-sided infinite sequences over* $\mathcal{A}$ is

$$\mathcal{A}^{\mathbb{N}} = \{x_0 x_1 x_2 \cdots : x_i \in \mathcal{A} \text{ for all } i \geq 0\}.$$

Definition 13.13. The *one-sided shift map* on $\mathcal{A}^{\mathbb{N}}$ is the function

$$\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$$

defined by

$$\sigma(x_0 x_1 x_2 \cdots) = x_1 x_2 x_3 \cdots .$$

Definition 13.14. A *one-sided symbolic dynamical system* is a pair (X, σ) where $X \subseteq \mathcal{A}^{\mathbb{N}}$ is a one-sided shift space given by a collection $\mathcal{F}$ of forbidden blocks and σ is the one-sided shift map.

Definition 13.15. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and let

$$\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$$

be the shift map. The set X is *shift invariant* if

$$\sigma(X) \subseteq X.$$

Equivalently, for every $x \in X$,

$$\sigma(x) \in X.$$

For a two-sided shift space, one often requires

$$\sigma(X) = X.$$

Module 14

Definition 14.1. Let X and Y be metric spaces. A function $f : X \to Y$ is *continuous* if for all $a \in X$ and any sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of points in X that converge to a, we have

$$\lim_{n\to\infty} f(x^{(n)}) = f(a).$$

Definition 14.2. Let $z \in [0, 1]$. A sequence $.z_0 z_1 \ldots$ is called a *binary expansion* of z if $z_i = 0$ or 1 for all $i = 0, 1, \ldots$ and

$$z = \sum_{i=0}^{\infty} z_i \cdot \frac{1}{2^{i+1}}.$$

A binary expansion is called *terminating* if it has only finitely many nonzero terms, and *nonterminating* otherwise.