1A

1.1 Definition (complex numbers)

- A *complex number* is an ordered pair (a, b), where $a, b \in \mathbb{R}$, but we write this as $a + bi$.
- The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- *Addition and multiplication* on $\mathbb{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

  where $a, b, c, d \in \mathbb{R}$.

1.3 Properties of complex arithmetic

- **commutativity:** $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$
- **associativity:** $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$
- **identities:** $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$
- **additive inverse:** for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$
- **multiplicative inverse:** for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$
- **distributive property:** $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$

1.5 Definition ($-\alpha$, subtraction, $1/\alpha$, division) Let $\alpha, \beta \in \mathbb{C}$.

- Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that

$$\alpha + (-\alpha) = 0.$$

- *Subtraction* on $\mathbb{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

- For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that

$$\alpha(1/\alpha) = 1.$$

- *Division* on $\mathbb{C}$ is defined by

$$\beta/\alpha = \beta(1/\alpha).$$

1.8 Definition (list, length) Suppose n is a nonnegative integer. A *list of length* n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \ldots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

1.10 Definition ($\mathbb{F}^n$) $\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}.$$

For $(x_1, \ldots, x_n) \in \mathbb{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the jth *coordinate* of $(x_1, \ldots, x_n)$.

1.12 Definition (addition in $\mathbb{F}^n$) Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

1.13 Commutativity of addition in $\mathbb{F}^n$ If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

1.14 Definition (0) Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0).$$

1.16 Definition (additive inverse in $\mathbb{F}^n$) For $x \in \mathbb{F}^n$, the *additive inverse* of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that

$$x + (-x) = 0.$$

In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$.

1.17 Definition (scalar multiplication in $\mathbb{F}^n$) The *product* of a number λ and a vector in $\mathbb{F}^n$ is computed by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n);$$

here $\lambda \in \mathbb{F}$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

1B

Definition 1.18 (addition, scalar multiplication)

- An *addition* on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

Definition 1.19 (vector space) A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- **Commutativity:** $u + v = v + u$ for all $u, v \in V$;
- **Associativity:** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{F}$;
- **Additive identity:** there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
- **Additive inverse:** for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;
- **Multiplicative identity:** $1v = v$ for all $v \in V$;
- **Distributive properties:** $a(u+v) = au+av$ and $(a+b)v = av+bv$ for all $a, b \in \mathbb{F}$ and $u, v \in V$.

Definition 1.20 (vector, point) Elements of a vector space are called *vectors* or *points.*

Definition 1.21 (real vector space, complex vector space)

- A vector space over $\mathbb{R}$ is called a *real vector space.*
- A vector space over $\mathbb{C}$ is called a *complex vector space.*

Notation 1.23 ($\mathbb{F}^S$)

- If S is a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$.
- For $f, g \in \mathbb{F}^S$, the *sum* $f + g \in \mathbb{F}^S$ is defined by
$$(f + g)(x) = f(x) + g(x)$$
for all $x \in S$.
- For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the *product* $\lambda f \in \mathbb{F}^S$ is defined by
$$(\lambda f)(x) = \lambda f(x)$$
for all $x \in S$.

Theorem 1.25 Unique additive identity
A vector space has a unique additive identity.

Theorem 1.26 Unique additive inverse
Every element in a vector space has a unique additive inverse.

Notation 1.27 $-v$, $w - v$
Let $v, w \in V$. Then:

- $-v$ denotes the additive inverse of v;
- $w - v$ is defined to be $w + (-v)$.

Notation 1.28 V
For the rest of the book, V denotes a vector space over $\mathbb{F}$.

Theorem 1.29 The number 0 times a vector
$0v = 0$ for every $v \in V$.

Theorem 1.30 A number times the vector 0
$a0 = 0$ for every $a \in \mathbb{F}$.

Theorem 1.31 The number -1 times a vector
$(-1)v = -v$ for every $v \in V$.

1C

Definition 1.32 (subspace)
A subset U of V is called a *subspace* of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Theorem 1.34 (conditions for a subspace)
A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

- **additive identity:** $0 \in U$
- **closed under addition:** $u, w \in U \Rightarrow u + w \in U$
- **closed under scalar multiplication:** $a \in \mathbb{F}$ and $u \in U \Rightarrow au \in U$

Definition 1.36 (sum of subsets)
Suppose $U_1, \ldots, U_m$ are subsets of V. The *sum* of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}.$$

Theorem 1.39 (sum of subspaces is the smallest containing subspace)
Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Definition 1.40 (direct sum)
Suppose $U_1, \ldots, U_m$ are subspaces of V.

- The sum $U_1 + \cdots + U_m$ is called a *direct sum* if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$.
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

Theorem 1.44 (condition for a direct sum)
Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$, is by taking each $u_j = 0$.

Theorem 1.45 (direct sum of two subspaces)
Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

2A

2.3 Definition (Linear Combination)
A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbb{F}$.

2.5 Definition (Span)
The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$, denoted $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}.$$

The span of the empty list () is defined to be $\{0\}$.

2.7 Span is the Smallest Containing Subspace
The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

2.8 Definition (Spans)
If $\operatorname{span}(v_1, \ldots, v_m) = V$, we say that $v_1, \ldots, v_m$ *span* V.

2.10 Definition (Finite-Dimensional Vector Space)
A vector space is called *finite-dimensional* if some list of vectors in it spans the space.

2.11 Definition (Polynomial, $\mathcal{P}(\mathbb{F})$)

- A function $p \colon \mathbb{F} \to \mathbb{F}$ is called a *polynomial* with coefficients in $\mathbb{F}$ if there exist $a_0, \ldots, a_m \in \mathbb{F}$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$
for all $z \in \mathbb{F}$.
- $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$.

2.12 Definition (Degree of a Polynomial, $\deg p$)

- A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have *degree m* if there exist scalars $a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that
$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$
for all $z \in \mathbb{F}$. If p has degree m, we write $\deg p = m$.
- The polynomial that is identically 0 is said to have degree $-\infty$.

2.13 Definition ($\mathcal{P}_m(\mathbb{F})$)
For m a nonnegative integer, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m.

2.15 Definition (Infinite-Dimensional Vector Space)
A vector space is called *infinite-dimensional* if it is not finite-dimensional.

2.17 Definition (Linearly Independent)

- A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that makes
$$a_1 v_1 + \cdots + a_m v_m = 0$$
is $a_1 = \cdots = a_m = 0$.

- The empty list () is also declared to be linearly independent.

2.19 Definition (Linearly Dependent)

- A list of vectors in V is called *linearly dependent* if it is not linearly independent.
- In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbb{F}$, not all 0, such that
$$a_1 v_1 + \cdots + a_m v_m = 0.$$

2.21 Linear Dependence Lemma
Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:

(a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;

(b) if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

2.23 Length of Linearly Independent List $\leq$ Length of Spanning List
In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

2.26 Corollary (Finite-Dimensional Subspaces)
Every subspace of a finite-dimensional vector space is finite-dimensional.

2B

2.27 Definition (Basis)
A *basis* of V is a list of vectors in V that is linearly independent and spans V.

2.29 Criterion for Basis
A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbb{F}$.

2.31 Spanning List Contains a Basis
Every spanning list in a vector space can be reduced to a basis of the vector space.

2.33 Linearly Independent List Extends to a Basis
Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

2.34 Every Subspace of V is Part of a Direct Sum Equal to V
Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

2C

2.35 Basis Length Does Not Depend on Basis
Any two bases of a finite-dimensional vector space have the same length.

2.36 Definition (Dimension), $\dim V$

- The *dimension* of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

2.38 Dimension of a Subspace
If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

2.39 Linearly Independent List of the Right Length is a Basis
Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

2.42 Spanning List of the Right Length is a Basis
Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

2.43 Dimension of a Sum
If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

3A

3.2 Definition (Linear Map)
A *linear map* from V to W is a function $T : V \to W$ with the following properties:

additivity

$$T(u + v) = Tu + Tv \text{ for all } u, v \in V;$$

homogeneity

$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbb{F} \text{ and all } v \in V.$$

3.3 Notation ($\mathcal{L}(V, W)$)
The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

3.5 Linear Maps and Basis of Domain
Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

3.6 Definition (Addition and Scalar Multiplication on $\mathcal{L}(V, W)$)
Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

3.7 $\mathcal{L}(V, W)$ is a Vector Space
With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

3.8 Definition (Product of Linear Maps)
If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$.

3.9 Algebraic Properties of Products of Linear Maps

associativity

$$(T_1T_2)T_3 = T_1(T_2T_3)$$

whenever T_1, T_2, and T_3 are linear maps such that the products make sense (meaning that T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1).

identity

$$TI = IT = T$$

whenever $T \in \mathcal{L}(V, W)$ (the first I is the identity map on V, and the second I is the identity map on W).

distributive properties

$$(S_1 + S_2)T = S_1T + S_2T \quad \text{and} \quad S(T_1 + T_2) = ST_1 + ST_2$$

whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

3.11 Linear Maps Take 0 to 0
Suppose T is a linear map from V to W. Then $T(0) = 0$.

3B

3.12 Definition (Null Space, null T)
For $T \in \mathcal{L}(V, W)$, the *null space* of T, denoted null T, is the subset of V consisting of those vectors that T maps to 0:

$$\text{null } T = \{v \in V : Tv = 0\}.$$

3.14 The Null Space is a Subspace
Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

3.15 Definition (Injective)
A function $T : V \to W$ is called *injective* if $Tu = Tv$ implies $u = v$.

3.16 Injectivity is Equivalent to Null Space Equals $\{0\}$
Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

3.17 Definition (Range)
For T a function from V to W, the *range* of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:

$$\text{range } T = \{Tv : v \in V\}.$$

3.19 The Range is a Subspace
If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.

3.20 Definition (Surjective)
A function $T : V \to W$ is called *surjective* if its range equals W.

3.22 Fundamental Theorem of Linear Maps
Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \text{null } T + \dim \text{range } T.$$

3.23 A Map to a Smaller Dimensional Space is Not Injective
Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

3.24 A Map to a Larger Dimensional Space is Not Surjective
Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

3.26 Homogeneous System of Linear Equations
A homogeneous system of linear equations with more variables than equations has nonzero solutions.

3.29 Inhomogeneous System of Linear Equations
An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

3C

3.30 Definition (Matrix, $A_{j,k}$)
Let m and n denote positive integers. An m-by-n *matrix* A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A. In other words, the first index refers to the row number and the second index refers to the column number.

3.32 Definition (Matrix of a Linear Map, $\mathcal{M}(T)$)
Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The *matrix of* T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$ is used.

3.35 Definition (Matrix Addition)
The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}.$$

In other words, $(A + C)_{j,k} = A_{j,k} + C_{j,k}$.

3.36 The Matrix of the Sum of Linear Maps
Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

3.37 Definition (Scalar Multiplication of a Matrix)
The *product of a scalar and a matrix* is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$.

3.38 The Matrix of a Scalar Times a Linear Map
Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda\mathcal{M}(T)$.

3.39 Notation ($\mathbb{F}^{m,n}$)
For m and n positive integers, the set of all m-by-n matrices with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{m,n}$.

3.40 $\dim \mathbb{F}^{m,n} = mn$
Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbb{F}^{m,n}$ is a vector space with dimension mn.

3.41 Definition (Matrix Multiplication)
Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then AC is defined to be the m-by-p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k}.$$

In other words, the entry in row j, column k, of AC is computed by taking row j of A and column k of C, multiplying together corresponding entries, and then summing.

3.43 The Matrix of the Product of Linear Maps
If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

3.44 Notation ($A_{j,\cdot}$, $A_{\cdot,k}$)
Suppose A is an m-by-n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

3.47 Entry of Matrix Product Equals Row Times Column
Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{j,k} = A_{j,\cdot}C_{\cdot,k}$$

for $1 \leq j \leq m$ and $1 \leq k \leq p$.

3.49 Column of Matrix Product Equals Matrix Times Column
Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k}$$

for $1 \leq k \leq p$.

3.52 Linear Combination of Columns
Suppose A is an m-by-n matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n-by-1 matrix. Then

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}.$$

In other words, Ac is a linear combination of the columns of A, with the scalars that multiply the columns coming from c.

3D

3.53 Definition (Invertible, Inverse)

- A linear map $T \in \mathcal{L}(V, W)$ is called *invertible* if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity map on V and TS equals the identity map on W.
- A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an *inverse* of T (note that the first I is the identity map on V and the second I is the identity map on W).

3.54 Inverse is Unique
An invertible linear map has a unique inverse.

3.55 Notation (T^{-1})
If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

3.56 Invertibility is Equivalent to Injectivity and Surjectivity
A linear map is invertible if and only if it is injective and surjective.

3.58 Definition (Isomorphism, Isomorphic)

- An *isomorphism* is an invertible linear map.
- Two vector spaces are called *isomorphic* if there is an isomorphism from one vector space onto the other one.

3.59 Dimension Shows Whether Vector Spaces are Isomorphic
Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

3.60 $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$ are Isomorphic
Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.

3.61 $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$
Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W).$$

3.62 Definition (Matrix of a Vector, $\mathcal{M}(v)$)
Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. The *matrix of* v with respect to this basis is the n-by-1 matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \ldots, c_n$ are the scalars such that

$$v = c_1 v_1 + \cdots + c_n v_n.$$

3.64 $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(v_k)$
Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Let $1 \leq k \leq n$. Then the kth column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$, equals $\mathcal{M}(v_k)$.

3.65 Linear Maps Act Like Matrix Multiplication
Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v).$$

3.67 Definition (Operator, $\mathcal{L}(V)$)

- A linear map from a vector space to itself is called an *operator*.
- The notation $\mathcal{L}(V)$ denotes the set of all operators on V. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

3.69 Injectivity is Equivalent to Surjectivity in Finite Dimensions
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is invertible;

(b) T is injective;

(c) T is surjective.

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5.2 Definition (Invariant Subspace)
Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant* under T if $u \in U$ implies $Tu \in U$.

5.5 Definition (Eigenvalue)
Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an *eigenvalue* of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

5.6 Equivalent Conditions to be an Eigenvalue
Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

(a) λ is an eigenvalue of T;

(b) $T - \lambda I$ is not injective;

(c) $T - \lambda I$ is not surjective;

(d) $T - \lambda I$ is not invertible.

Note: Recall that $I \in \mathcal{L}(V)$ is the identity operator defined by $Iv = v$ for all $v \in V$.

5.7 Definition (Eigenvector)
Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

5.10 Linearly Independent Eigenvectors
Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

5.13 Number of Eigenvalues
Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

5.14 Definition ($T|_U$ and T/U)
Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T.

- The *restriction operator* $T|_U \in \mathcal{L}(U)$ is defined by

$$T|_U(u) = Tu$$

  for $u \in U$.

- The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U$$

  for $v \in V$.

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5.16 Definition (T^m)
Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- T^m is defined by
$$T^m = \underbrace{T \cdots T}_{m \text{ times}}.$$
- T^0 is defined to be the identity operator I on V.
- If T is invertible with inverse T^{-1}, then T^{-m} is defined by
$$T^{-m} = (T^{-1})^m.$$

5.17 Definition ($p(T)$)
Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$
for $z \in \mathbb{F}$. Then $p(T)$ is the operator defined by
$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

5.19 Definition (Product of Polynomials)
If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by
$$(pq)(z) = p(z)q(z)$$
for $z \in \mathbb{F}$.

5.20 Multiplicative Properties
Suppose $p, q \in \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(V)$. Then

(a) $(pq)(T) = p(T)q(T)$;

(b) $p(T)q(T) = q(T)p(T)$.

Note: Part (a) holds because when expanding a product of polynomials using the distributive property, it does not matter whether the symbol is z or T.

5.21 Operators on Complex Vector Spaces Have an Eigenvalue
Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

5.22 Definition (Matrix of an Operator, $\mathcal{M}(T)$)
Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. The *matrix of* T with respect to this basis is the n-by-n matrix
$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$
whose entries $A_{j,k}$ are defined by
$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$
If the basis is not clear from the context, then the notation $\mathcal{M}(T, (v_1, \ldots, v_n))$ is used.

5.24 Definition (Diagonal of a Matrix)
The *diagonal* of a square matrix consists of the entries along the line from the upper left corner to the bottom right corner.

5.25 Definition (Upper-Triangular Matrix)
A matrix is called *upper triangular* if all the entries below the diagonal equal 0.

5.26 Conditions for Upper-Triangular Matrix
Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent:

(a) the matrix of T with respect to $v_1, \dots, v_n$ is upper triangular;

(b) $Tv_j \in \operatorname{span}(v_1, \dots, v_j)$ for each $j = 1, \dots, n$;

(c) $\operatorname{span}(v_1, \dots, v_j)$ is invariant under T for each $j = 1, \dots, n$.

5.27 Over $\mathbb{C}$, Every Operator Has an Upper-Triangular Matrix
Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

5.30 Determination of Invertibility from Upper-Triangular Matrix
Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then T is invertible if and only if all the entries on the diagonal of that upper-triangular matrix are nonzero.

5.32 Determination of Eigenvalues from Upper-Triangular Matrix
Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

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5.34 Definition (Diagonal Matrix)
A *diagonal matrix* is a square matrix that is 0 everywhere except possibly along the diagonal.

5.36 Definition (Eigenspace, $E(\lambda, T)$)
Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The *eigenspace* of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \text{null}(T - \lambda I).$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

5.38 Sum of Eigenspaces is a Direct Sum
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore,

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

5.39 Definition (Diagonalizable)
An operator $T \in \mathcal{L}(V)$ is called *diagonalizable* if the operator has a diagonal matrix with respect to some basis of V.

5.41 Conditions Equivalent to Diagonalizability
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

(a) T is diagonalizable;

(b) V has a basis consisting of eigenvectors of T;

(c) there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V, each invariant under T, such that

$$V = U_1 \oplus \cdots \oplus U_n;$$

(d) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$;

(e) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

5.44 Enough Eigenvalues Implies Diagonalizability
If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

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6.2 Definition (dot product)
For $x, y \in \mathbb{R}^n$, the *dot product* of x and y, denoted $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

6.3 Definition (inner product)
An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbb{F}$ and has the following properties:

- **positivity**: $\langle v, v\rangle \geq 0$ for all $v \in V$;
- **definiteness**: $\langle v, v\rangle = 0$ if and only if $v = 0$;
- **additivity in first slot**: $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;
- **homogeneity in first slot**: $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbb{F}$ and all $u, v \in V$;
- **conjugate symmetry**: $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

6.5 Definition (inner product space)
An *inner product space* is a vector space V along with an inner product on V.

6.7 Basic properties of an inner product

(a) For each fixed $u \in V$, the function that takes v to $\langle v, u\rangle$ is a linear map from V to $\mathbb{F}$.

(b) $\langle 0, u\rangle = 0$ for every $u \in V$.

(c) $\langle u, 0\rangle = 0$ for every $u \in V$.

(d) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.

(e) $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$ for all $\lambda \in \mathbb{F}$ and $u, v \in V$.

6.8 Definition (norm, $\|v\|$)
For $v \in V$, the *norm* of v, denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

6.10 Basic properties of the norm
Suppose $v \in V$.

(a) $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda|\|v\|$ for all $\lambda \in \mathbb{F}$.

6.11 Definition (orthogonal)
Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v\rangle = 0$.

6.12 Orthogonality and 0

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

6.13 Pythagorean Theorem
Suppose u and v are orthogonal vectors in V. Then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

6.14 An orthogonal decomposition
Suppose $u, v \in V$, with $v \neq 0$. Set $c = \frac{\langle u,v\rangle}{\|v\|^2}$ and $w = u - \frac{\langle u,v\rangle}{\|v\|^2}v$. Then

$$\langle w, v\rangle = 0 \quad \text{and} \quad u = cv + w.$$

6.15 Cauchy–Schwarz Inequality
Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

6.18 Triangle Inequality
Suppose $u, v \in V$. Then

$$\|u+v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of u, v is a nonnegative multiple of the other.

6.22 Parallelogram Equality
Suppose $u, v \in V$. Then

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

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6.23 Definition (orthonormal)

- A list of vectors is called *orthonormal* if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In other words, a list $e_1, \ldots, e_m$ of vectors in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

6.25 The norm of an orthonormal linear combination
If $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{F}$.

6.26 An orthonormal list is linearly independent
Every orthonormal list of vectors is linearly independent.

6.27 Definition (orthonormal basis)
An *orthonormal basis* of V is an orthonormal list of vectors in V that is also a basis of V.

6.28 An orthonormal list of the right length is an orthonormal basis
Every orthonormal list of vectors in V with length $\dim V$ is an orthonormal basis of V.

6.30 Writing a vector as linear combination of orthonormal basis
Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2.$$

6.31 Gram–Schmidt Procedure
Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $e_1 = v_1/\|v_1\|$. For $j = 2, \ldots, m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\|v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}\|}.$$

Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$$

for $j = 1, \ldots, m$.

6.34 Existence of orthonormal basis
Every finite-dimensional inner product space has an orthonormal basis.

6.35 Orthonormal list extends to orthonormal basis
Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

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6.45 Definition (orthogonal complement, $U^\perp$)
If U is a subset of V, then the *orthogonal complement* of U, denoted $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{v \in V : \langle v, u\rangle = 0 \text{ for every } u \in U\}.$$

6.46 Basic properties of orthogonal complement

(a) If U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{0\}^\perp = V$.

(c) $V^\perp = \{0\}$.

(d) If U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

(e) If U and W are subsets of V and $U \subseteq W$, then $W^\perp \subseteq U^\perp$.

6.47 Direct sum of a subspace and its orthogonal complement
Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^\perp.$$

6.50 Dimension of the orthogonal complement
Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^\perp = \dim V - \dim U.$$

6.51 The orthogonal complement of the orthogonal complement
Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^\perp)^\perp.$$

7.2 Definition (adjoint, T^*)
Suppose $T \in \mathcal{L}(V, W)$. The *adjoint* of T is the function $T^* : W \to V$ such that

$$\langle Tv, w\rangle = \langle v, T^*w\rangle$$

for every $v \in V$ and every $w \in W$.

7.5 The adjoint is a linear map
If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

7.6 Properties of the adjoint

(a) $(S + T)^* = S^* + T^*$ for all $S, T \in \mathcal{L}(V, W)$;

(b) $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$;

(c) $(T^*)^* = T$ for all $T \in \mathcal{L}(V, W)$;

(d) $I^* = I$, where I is the identity operator on V;

(e) $(ST)^* = T^*S^*$ for all $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$ (here U is an inner product space over $\mathbb{F}$).

7.7 Null space and range of T^*
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) null $T^* = (\text{range } T)^\perp$;

(b) range $T^* = (\text{null } T)^\perp$;

(c) null $T = (\text{range } T^*)^\perp$;

(d) range $T = (\text{null } T^*)^\perp$.

7.8 Definition (conjugate transpose)
The *conjugate transpose* of an m-by-n matrix is the n-by-m matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.

7.10 The matrix of T^*
Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Then

$$\mathcal{M}(T^*, (f_1, \ldots, f_m), (e_1, \ldots, e_n))$$

is the conjugate transpose of

$$\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_m)).$$

7.11 Definition (self-adjoint)
An operator $T \in \mathcal{L}(V)$ is called *self-adjoint* if $T = T^*$. In other words, $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle$$

for all $v, w \in V$.

7.13 Eigenvalues of self-adjoint operators are real
Every eigenvalue of a self-adjoint operator is real.

7.14 Over $\mathbb{C}$, Tv is orthogonal to v for all v only for the 0 operator
Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Suppose

$$\langle Tv, v \rangle = 0$$

for all $v \in V$. Then $T = 0$.

7.15 Over $\mathbb{C}$, $\langle Tv, v \rangle$ is real for all v only for self-adjoint operators
Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then T is self-adjoint if and only if

$$\langle Tv, v \rangle \in \mathbb{R}$$

for every $v \in V$.

7.16 If $T = T^*$ and $\langle Tv, v \rangle = 0$ for all v, then $T = 0$
Suppose T is a self-adjoint operator on V such that

$$\langle Tv, v \rangle = 0$$

for all $v \in V$. Then $T = 0$.

7.18 Definition (normal)

- An operator on an inner product space is called *normal* if it commutes with its adjoint.
- In other words, $T \in \mathcal{L}(V)$ is normal if
$$TT^* = T^*T.$$

7.20 T is normal if and only if $\|Tv\| = \|T^*v\|$ for all v
An operator $T \in \mathcal{L}(V)$ is normal if and only if

$$\|Tv\| = \|T^*v\|$$

for all $v \in V$.

7.21 For T normal, T and T^* have the same eigenvectors
Suppose $T \in \mathcal{L}(V)$ is normal and $v \in V$ is an eigenvector of T with eigenvalue λ. Then v is also an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

7.22 Orthogonal eigenvectors for normal operators
Suppose $T \in \mathcal{L}(V)$ is normal. Then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

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7.24 Complex Spectral Theorem
Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

7.26 Invertible quadratic expressions
Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 < 4c$. Then

$$T^2 + bT + cI$$

is invertible.

7.27 Self-adjoint operators have eigenvalues
Suppose $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.

7.28 Self-adjoint operators and invariant subspaces
Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then

(a) $U^\perp$ is invariant under T;

(b) $T|_U \in \mathcal{L}(U)$ is self-adjoint;

(c) $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

7.29 Real Spectral Theorem
Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is self-adjoint.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.